从黑天鹅事件到量子Griffiths奇异性

--统计物理教会我们换一个角度看世界

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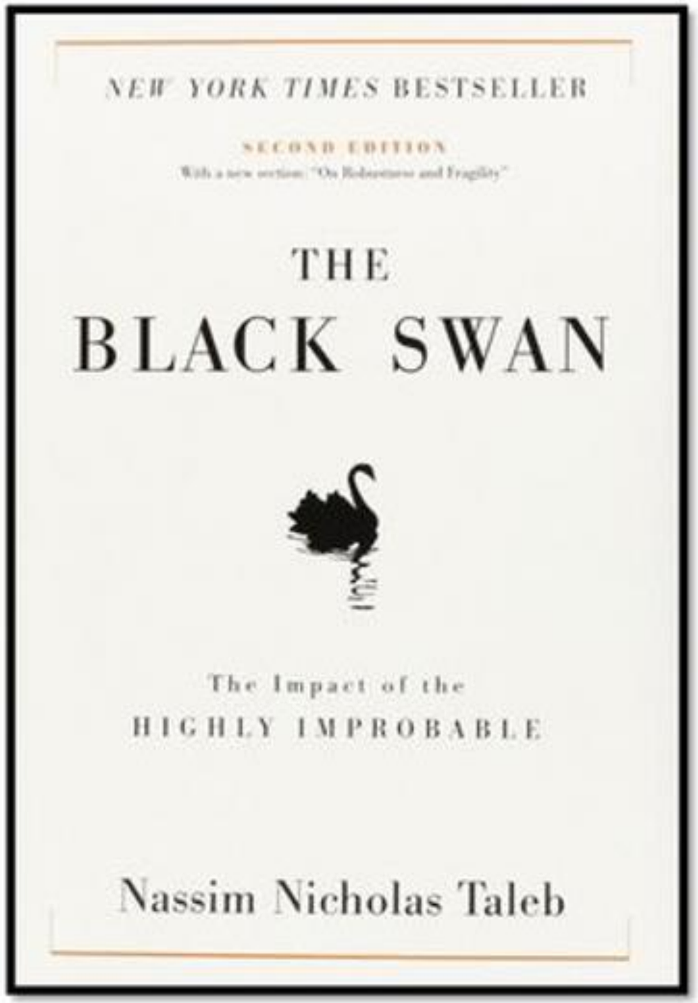
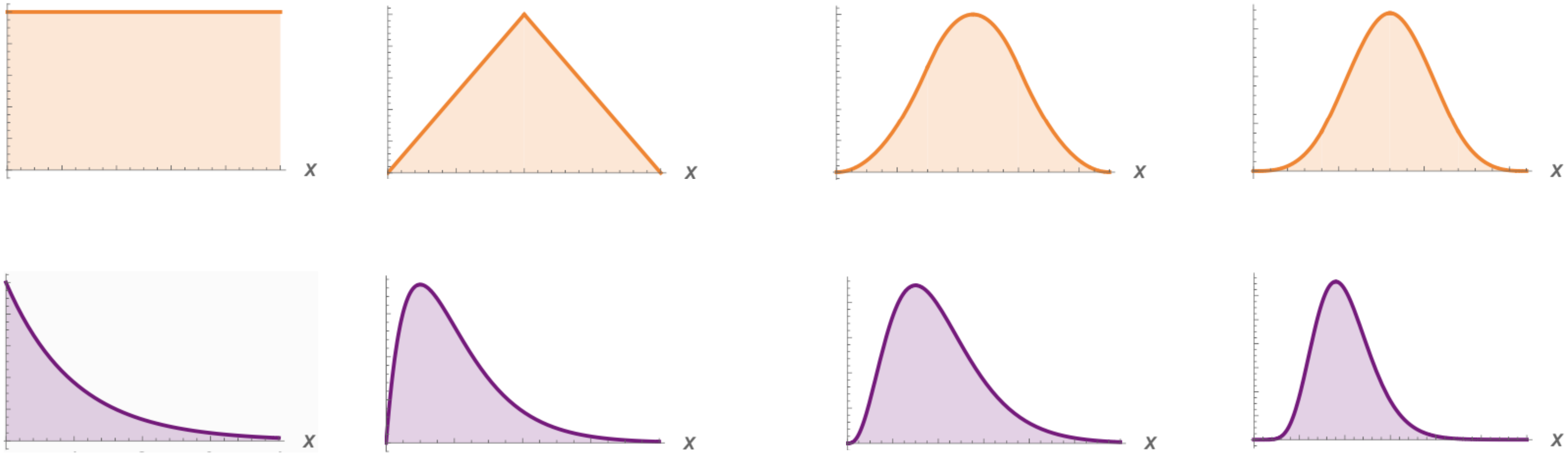
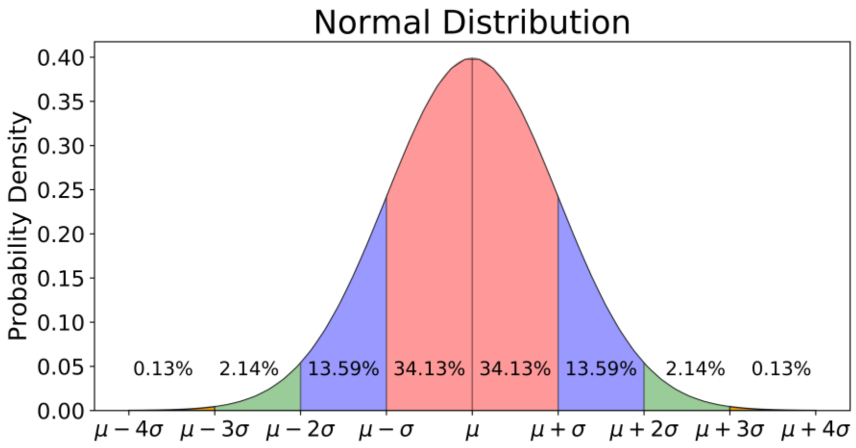
Beijing, July 29, 2020

Rare event associated with fat tail distribution



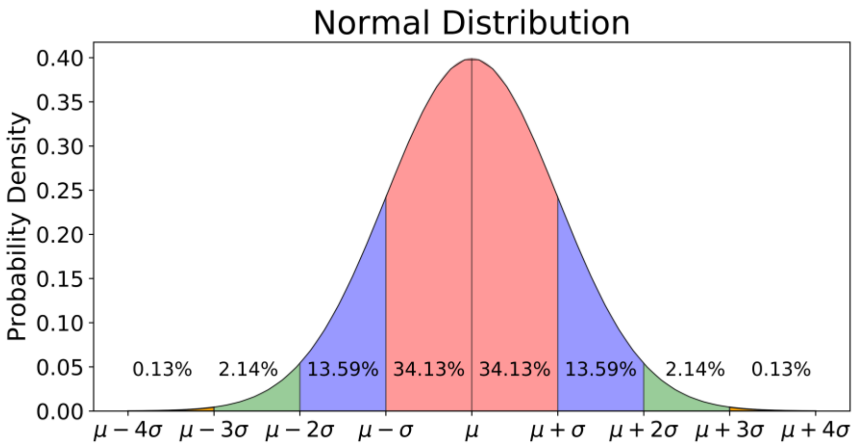
Gaussian distribution and Central limit theorem





In probability theory, the central limit theorem (CLT) establishes that, in some situations, when independent random variables are added, their properly normalized sum tends toward a normal distribution (informally a bell curve) even if the original variables themselves are not normally distributed.

Central limit theorem from the aspect of stable law

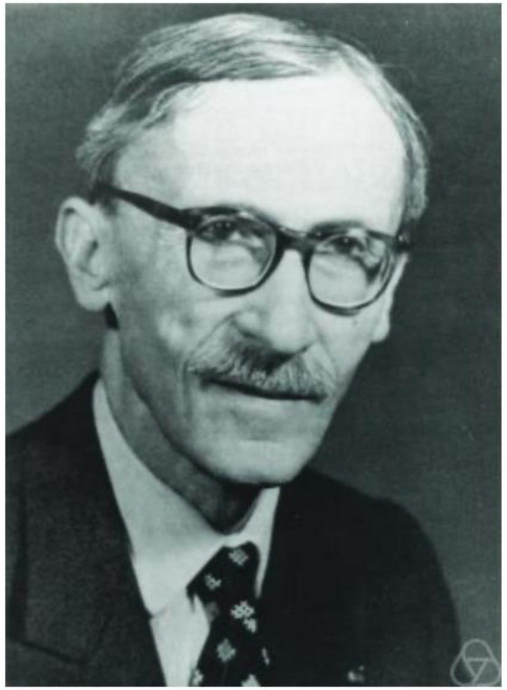


$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2} \right)$$

The normal distribution is the fixed-point function under iteration.

$$P(x) \sim \frac{C}{|x|^{1+\mu}} \qquad \text{for } x \to \pm\infty \qquad 0 < \mu < 2$$

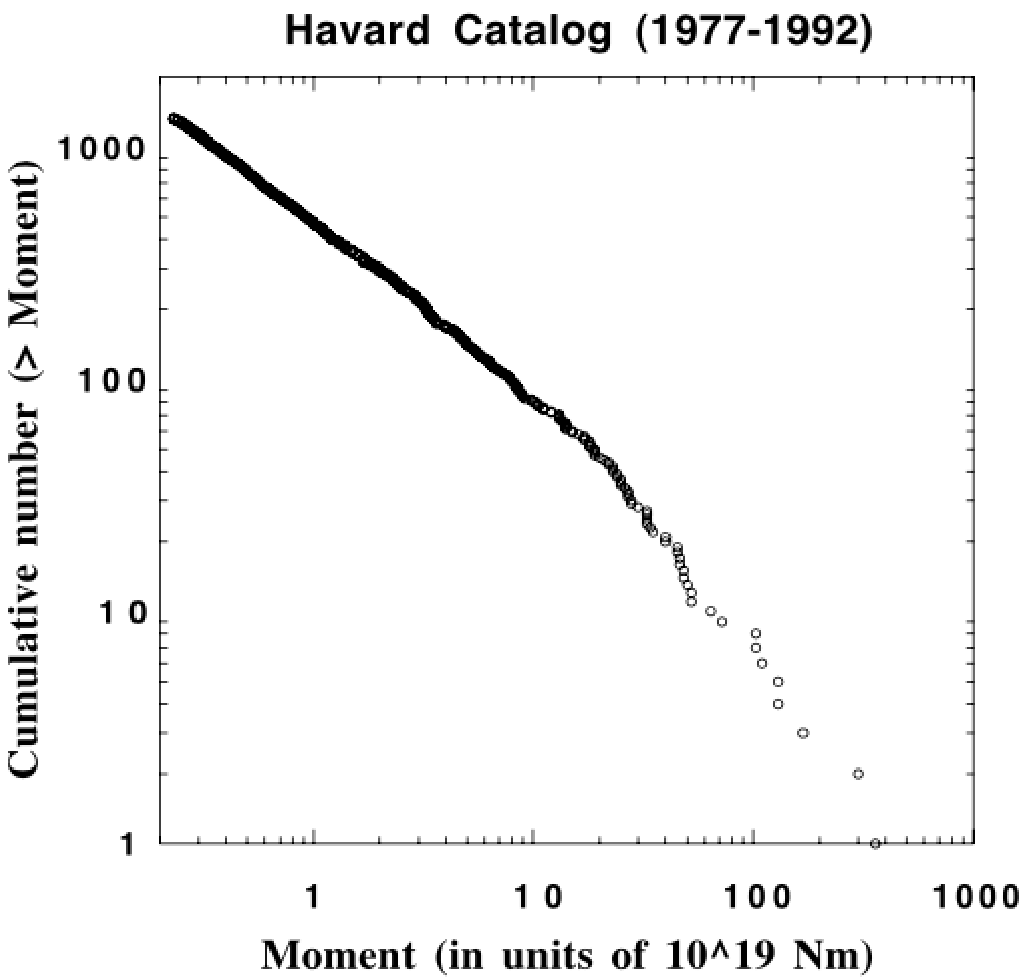
The power law distribution is also stable under iteration.



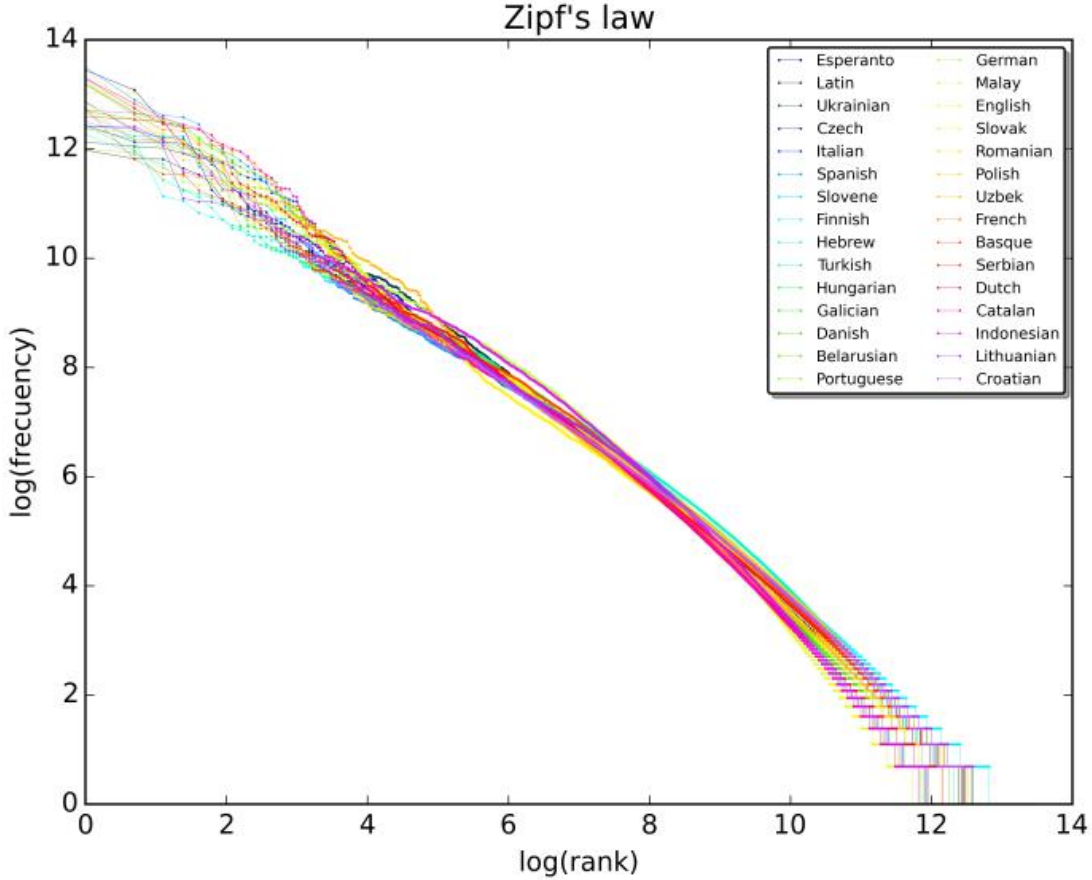
Paul Lévy, 1886-1971

Paul Lévy formulated the generalized CLT.

Power law distribution

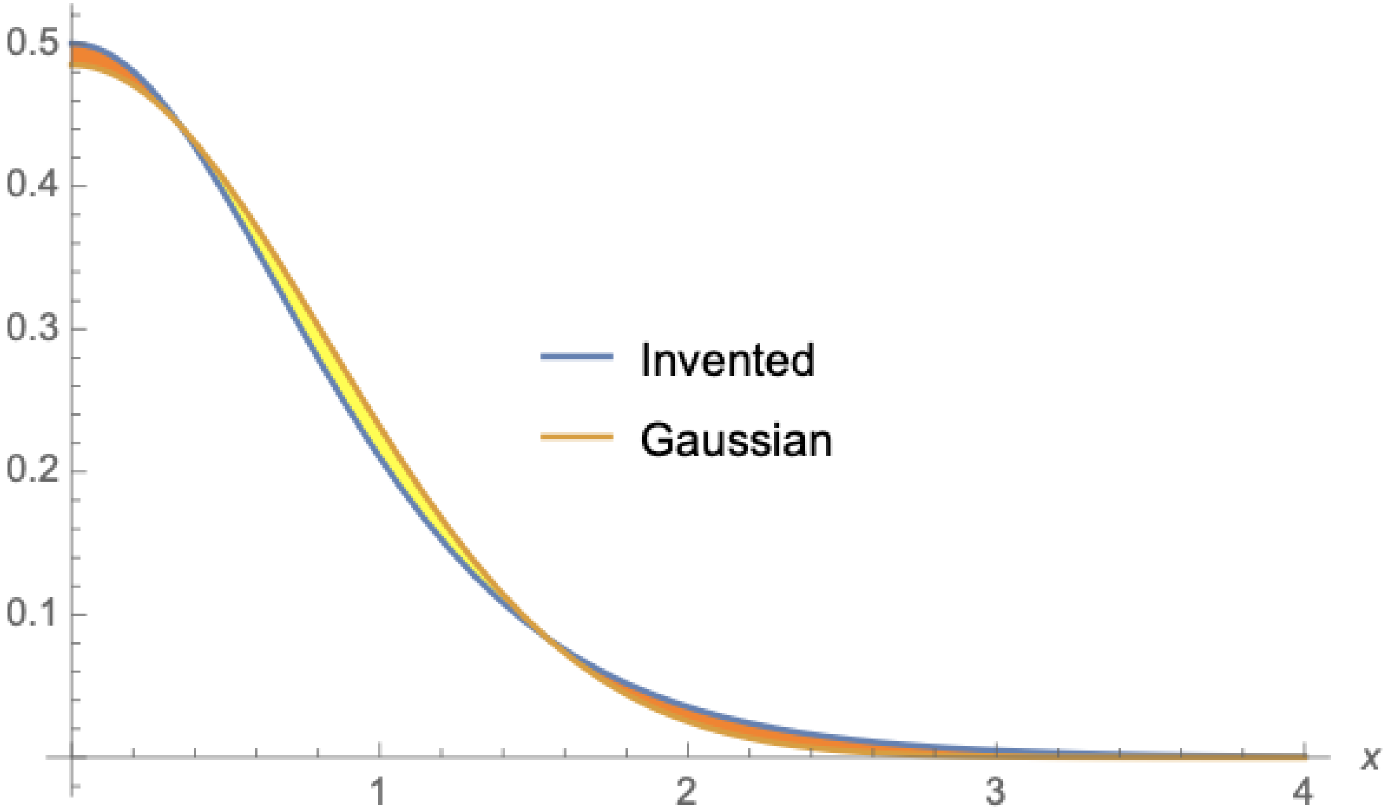


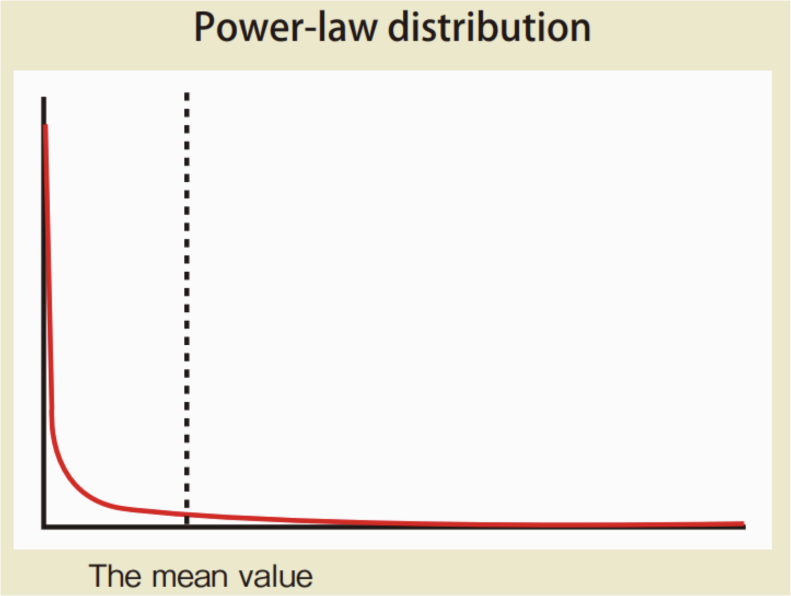
D. Sornette, et al., Rank-ordering statistics of extreme events: application to the distribution of large earthquakes, J. Geophys. Res. 101, 13883 (1996).

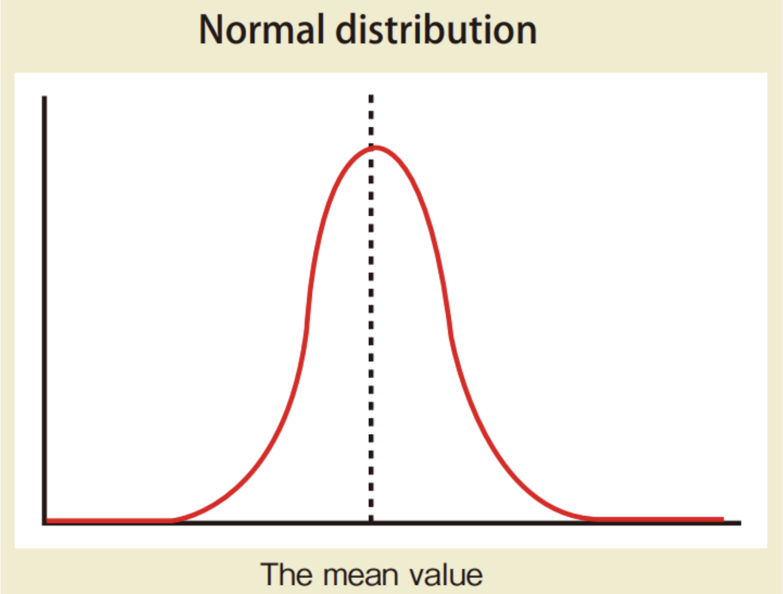


Word frequency versus rank.
(**Zipf–Mandelbrot law**)
From Wikipedia.

Power law distribution



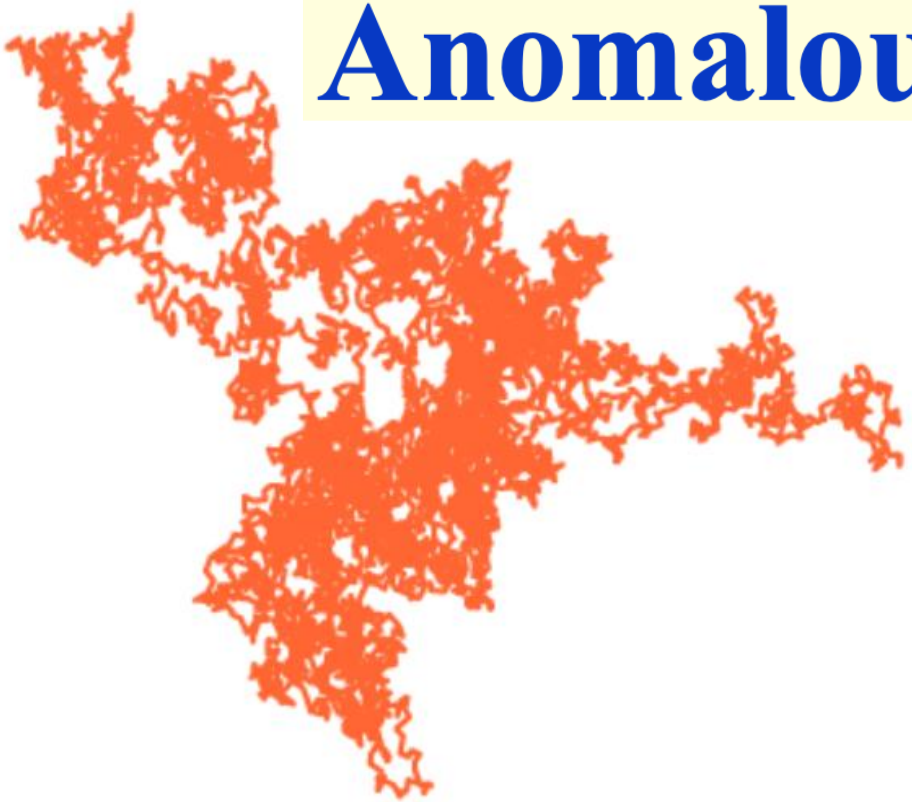


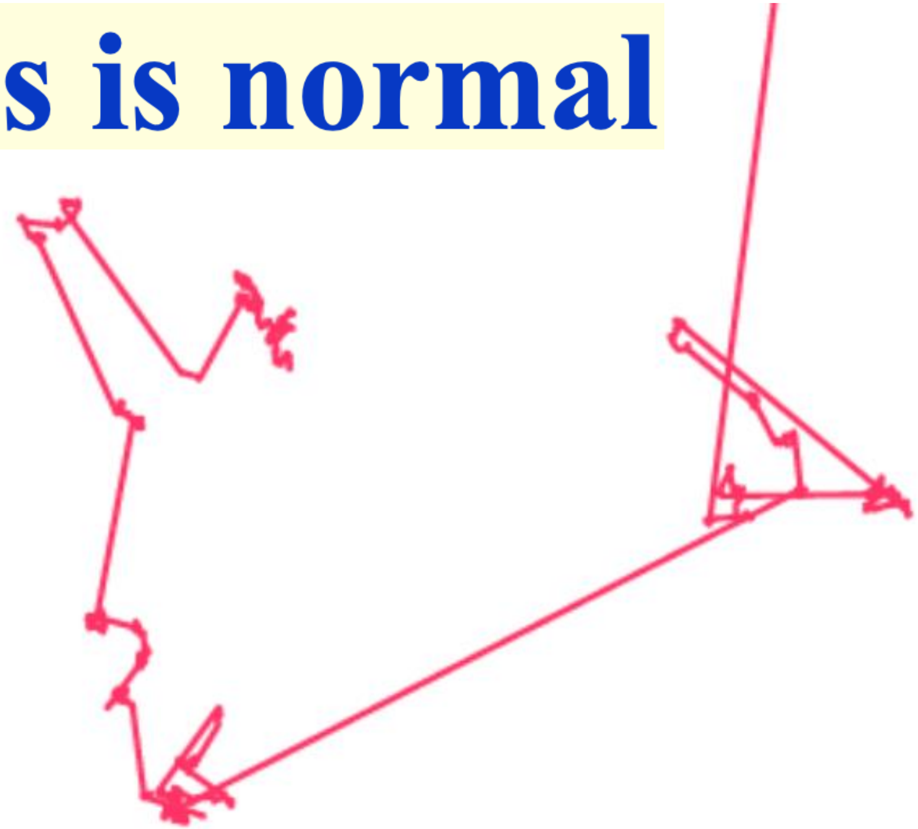


Rare event associated with fat tail and high peaks.

Manifestation of abnormal distribution → random process

Anomalous is normal





$$< r^2(t) >\sim Kt$$

$$< r^2(t) >\sim t^{\alpha}$$

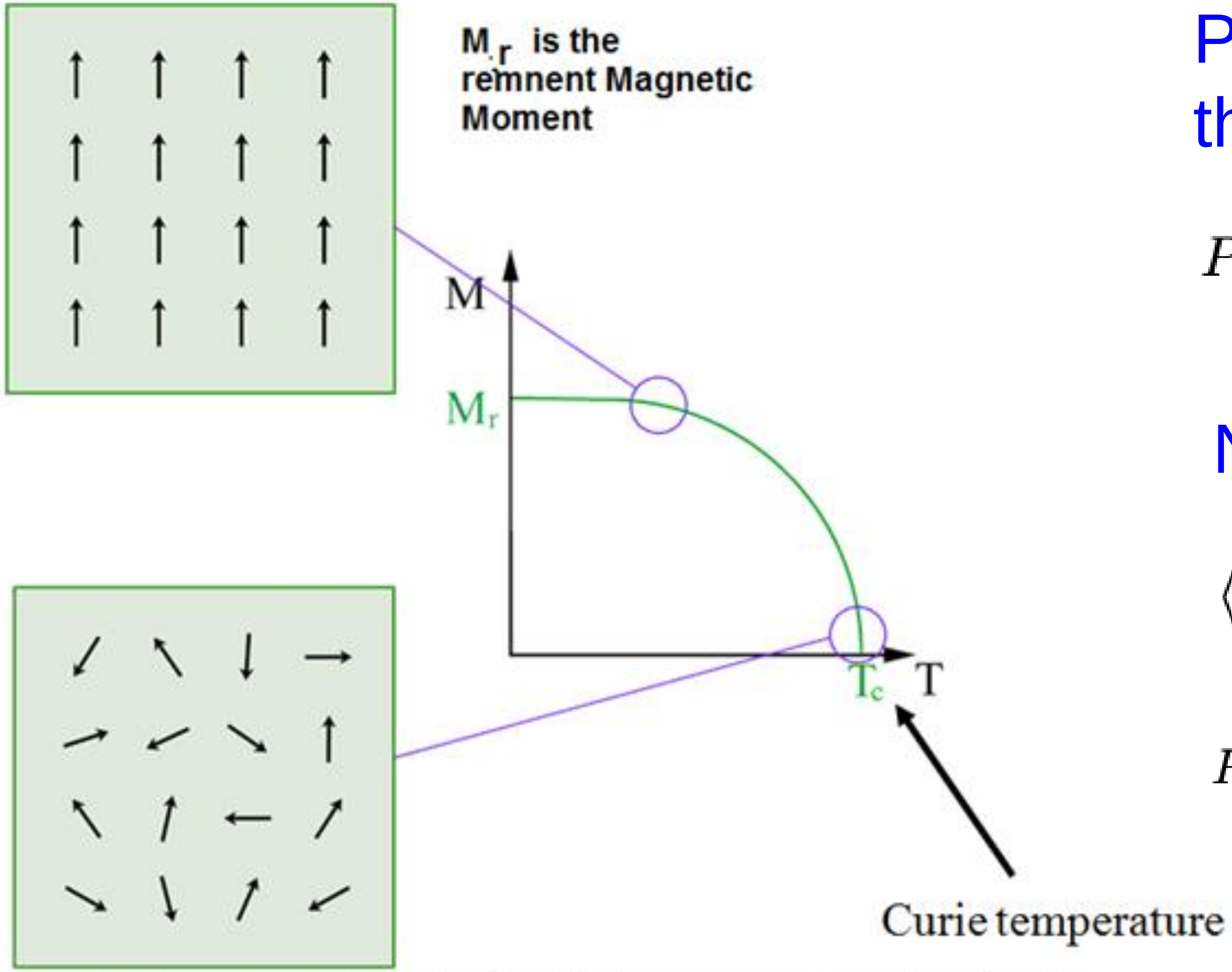
$\alpha < 1$ Subdiffusion (dispersive)
$\alpha > 1$ Superdiffusion

Brownian motion is associated with Normal distribution.

Anomalous diffusion is associated with Lévy distribution.

Gaussian distribution, deviation and relation to statistical physics

Take the spin system as an example.



In Ferromagnetic materials dipole moments are aligned upto 1000K.Only above Curie Temperature that alignment get disrupted.

Paramagnetic phase:
the magnetization is Gaussian

$$P(M) \sim \frac{1}{\sqrt{2\pi\chi L^d}} e^{-M^2/2\chi L^d} \qquad M = \sum_{i=1}^{L^d} S_i$$

Near the Curie point:

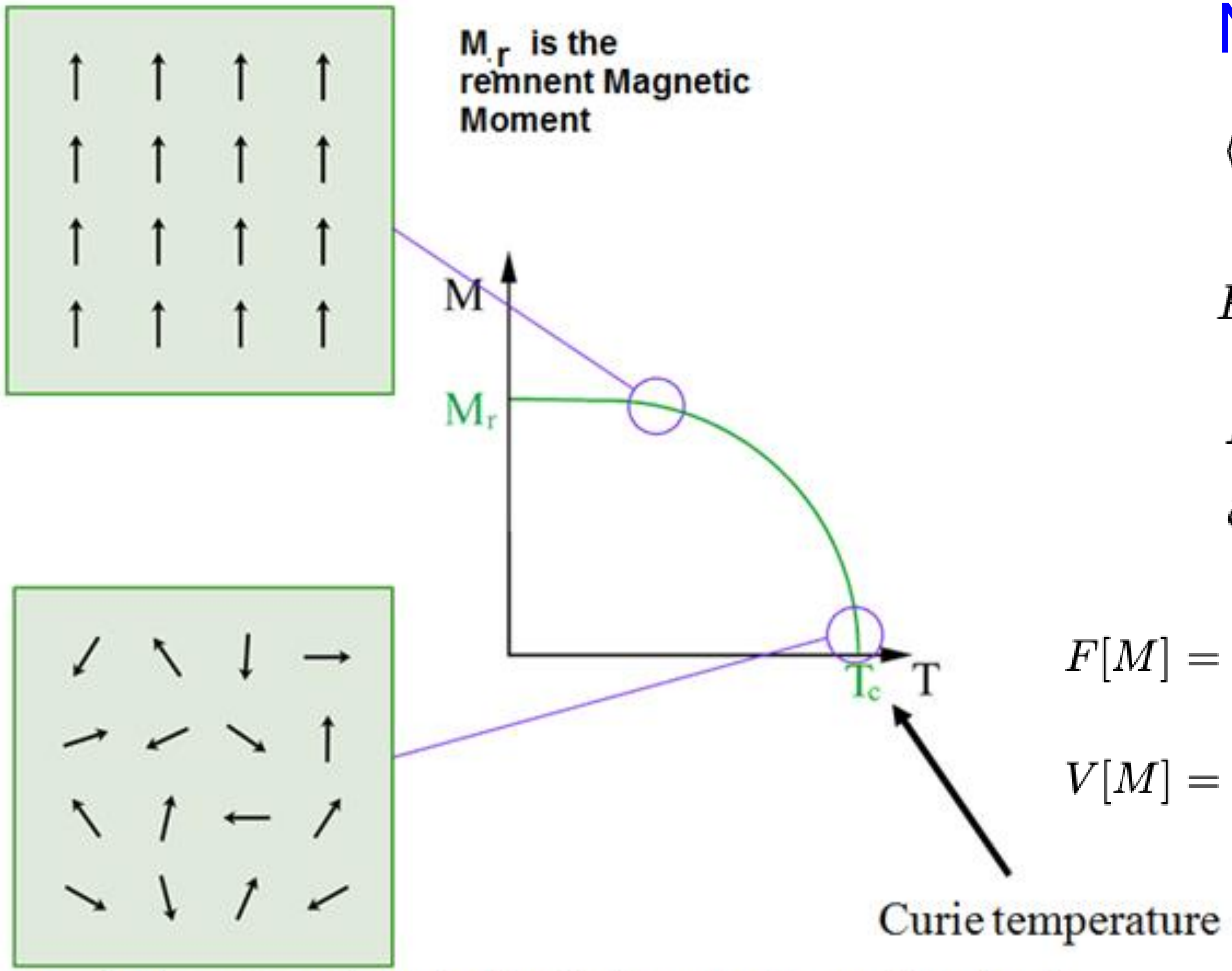
$$\langle S_0 S_r \rangle \sim \frac{1}{r^{d-2+\eta}}$$

$$P(M) = \frac{1}{L^{d\nu}} f\left(\frac{M}{(L^d)^\nu}\right) \qquad \nu = (d+2-\eta)/2d$$

$$F \sim L^d |T - T_c|^{2-\alpha}$$

$$\xi \sim |T - T_c|^{-\nu_{th}}$$

Take the spin system as an example – Ginzburg Landau theory



In Ferromagnetic materials dipole moments are aligned upto 1000K.Only above Curie Temperature that alignment get disrupted.

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$$F \sim L^d |T - T_c|^{2-\alpha}$$

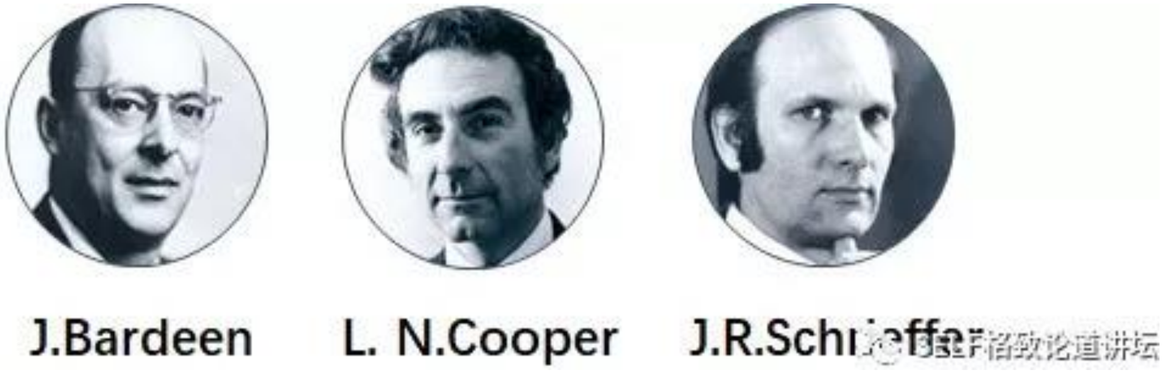
$$\xi \sim |T - T_c|^{-\nu_{th}}$$

$$F[M] = \int d^d\mathbf{x} \left\{ \frac{1}{2} [\nabla M(\mathbf{x})]^2 + V[M(\mathbf{x})] \right\} ,$$

$$V[M] = \frac{1}{2} r(T) M^2 + \frac{\lambda}{4} M^4 - H M \ ; \quad r(T) = r_0 (T - T_c) \ ,$$

$$M_0(T) = \begin{cases} 0 & \text{for } T > T_c \\ \sqrt{r_0/\lambda} (T_c - T)^{1/2} & \text{for } T < T_c \end{cases}$$

Deviation from the Ginzburg-Landau theory: manifestation in 2D superconductors



Phys. Rev. 108, 1175 (1957).

罗会仟撰文，《108年10个诺贝尔奖，火爆的超导到底能带来啥？》
来自返朴微信公众号 https://mp.weixin.qq.com/s/_olNtOuCX5N55L7tbxyoGg

Nonmagnetic disorder effect on s-wave superconductivity

J. Phys. Chem. Solids Pergamon Press 1959. Vol. 11. pp. 26–30. Printed in Great Britain.

THEORY OF DIRTY SUPERCONDUCTORS

P. W. ANDERSON

Bell Telephone Laboratories, Murray Hill, New Jersey

(*Received* 3 *March* 1959)

Abstract—A B.C.S. type of theory (see BARDEEN, COOPER and SCHREIFFER, *Phys. Rev.* **108,** 1175 (1957)) is sketched for very dirty superconductors, where elastic scattering from physical and chemical impurities is large compared with the energy gap. This theory is based on pairing each one-electron state with its exact time reverse, a generalization of the *k* up, −*k* down pairing of the B.C.S. theory which is independent of such scattering. Such a theory has many qualitative and a few quantitative points of agreement with experiment, in particular with specific-heat data, energy-gap measurements, and transition-temperature versus impurity curves. Other types of pairing which have been suggested are not compatible with the existence of dirty superconductors.

Other types of pairing which have been suggested are not compatible with the existence of dirty superconductors.

SOVIET PHYSICS JETP VOLUME 35(8), NUMBER 6 JUNE, 1959

ON THE THEORY OF SUPERCONDUCTING ALLOYS

I. THE ELECTRODYNAMICS OF ALLOYS AT ABSOLUTE ZERO

A. A. ABRIKOSOV and L. P. GOR'KOV

Institute for Physics Problems, Academy of Sciences, U.S.S.R.

Submitted to JETP editor July 16, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) 35, 1558-1571 (December, 1958)

L. P. Gor'kov, Theory of Superconducting Alloys, Chapter 5 in Superconductivity: Conventional and Unconventional (2008).

Introduction to the Cooper pair breaking mechanisms

1. Paramagnetic Effects:

1.a) Zeeman splitting due to external magnetic field: Clogston-Chandrasekhar limit (so-called Pauli limit)

A. M. Clogston, Phys. Rev. Lett. 9, 266 (1962).

B. S. Chandrasekhar, Appl. Phys. Lett. 1, 7 (1962).

$$g\mu_B B_p = 3.52 k_B T_{c,0}$$

$$ln\left(\frac{T_{c,0}}{T_c}\right) = Re\left[\psi\left(\frac{1}{2} + \frac{i\mu_B B_c}{2\pi k_B T_c}\right) - \psi\left(\frac{1}{2}\right)\right]$$

1.b) Paramagnetic impurities:

A. A. Abrikosov, L.P.Gor'kov,Sov.Phys.JETP12,1243(1960).

K. Maki, Physics 1, 21 (1964)

P. G. de Gennes, Phys. Kondensierten Materie 3, 79 (1964).

$$ln\left(\frac{T_{c,0}}{T_c}\right) = \psi\left(\frac{1}{2} + \frac{1}{2\pi k_B T_c \tau_S}\right) - \psi\left(\frac{1}{2}\right)$$

τ_S is the pair breaking time due to scattering

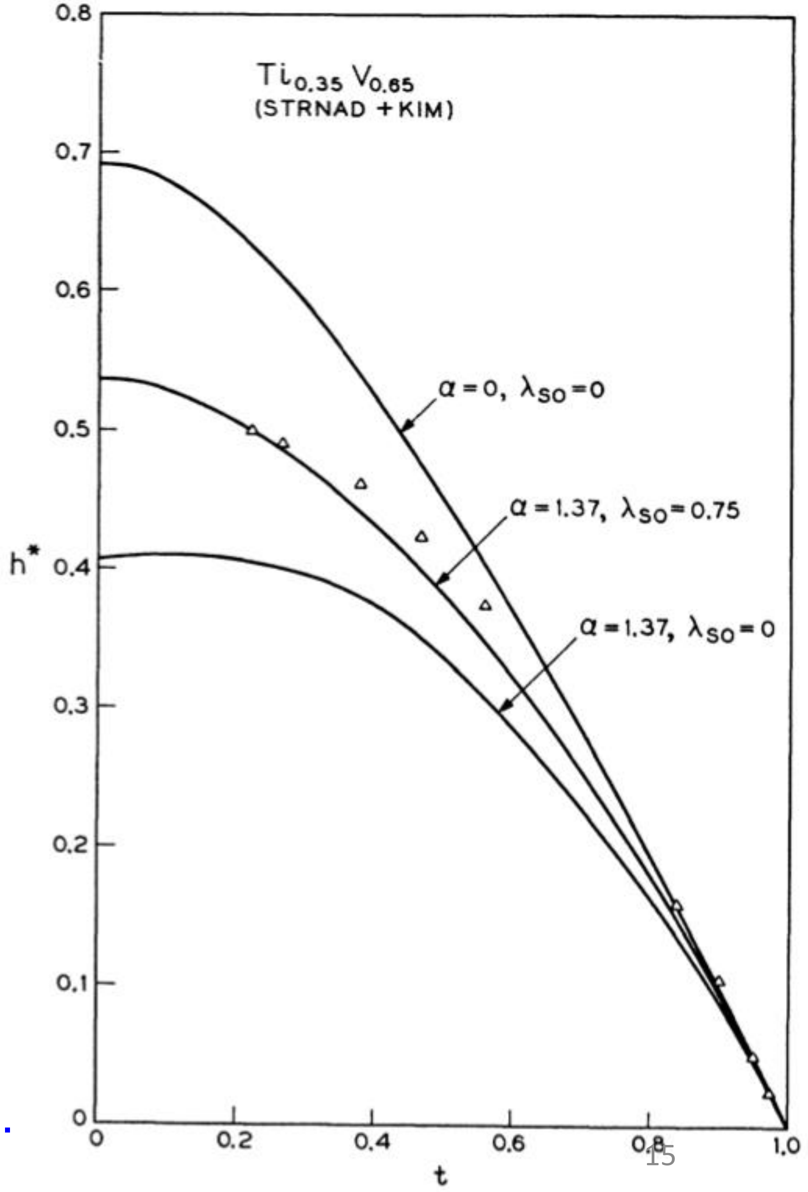
Introduction to the Cooper pair breaking mechanisms

2. Orbital Effects: (also include the paramagnetic effect and the spin orbital scattering)

Werthamer, Helfand and Hohenberg have generalized AG's theory to include the orbital effect, the para-magnetic effect and the spin-orbital scattering effect.

full solution

$$ln\left(\frac{1}{t}\right) = \left(\frac{1}{2} + \frac{i\lambda_{so}}{4\gamma}\right)\psi\left(\frac{1}{2} + \frac{h + \frac{1}{2}\lambda_{so} + i\gamma}{2t}\right)$$
$$+\left(\frac{1}{2} - \frac{i\lambda_{so}}{4\gamma}\right)\psi\left(\frac{1}{2} + \frac{h+\frac{1}{2}\lambda_{so}-i\gamma}{2t}\right) - \psi\left(\frac{1}{2}\right)$$



N. R. Werthamer, E. Helfand, and P. C. Hohenberg, Phys. Rev. 147, 295 (1966).
K. Maki and T. Tsuneto, Progr. Theor. Phys. (Kyoto) 31, 945 (1964).

Introduction to the Cooper pair breaking mechanisms

3. Upper critical field in Layered superconductors:

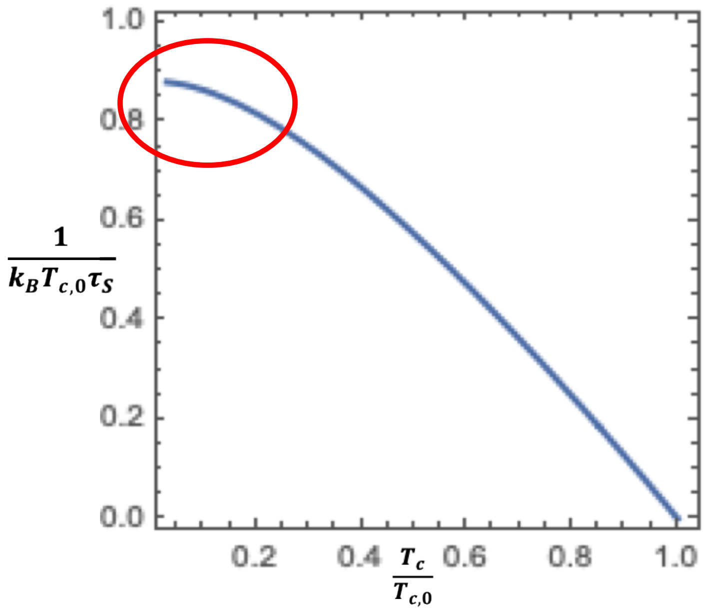
R. A. Klemm, A. Luther, and M. R. Beasley, Phys. Rev. B 12, 877 (1975).

Neglect the interlayer coupling, and when spin-flip scattering dominates:

$$ln\left(\frac{T_{c,0}}{T_c}\right) = \psi\left(\frac{1}{2} + \frac{3\tau_{SO}}{2\hbar}\frac{\mu_B^2 B_c^2}{2\pi k_B T_c}\right) - \psi\left(\frac{1}{2}\right)$$

Pauli limit can be exceeded in large spin-orbital scattering systems.

$$ln\left(\frac{T_{c,0}}{T_c}\right) = \psi\left(\frac{1}{2} + \frac{1}{2\pi k_B T_c \tau_S}\right) - \psi\left(\frac{1}{2}\right)$$



Abrikosov and Gor'kov's standard pair breaking theory

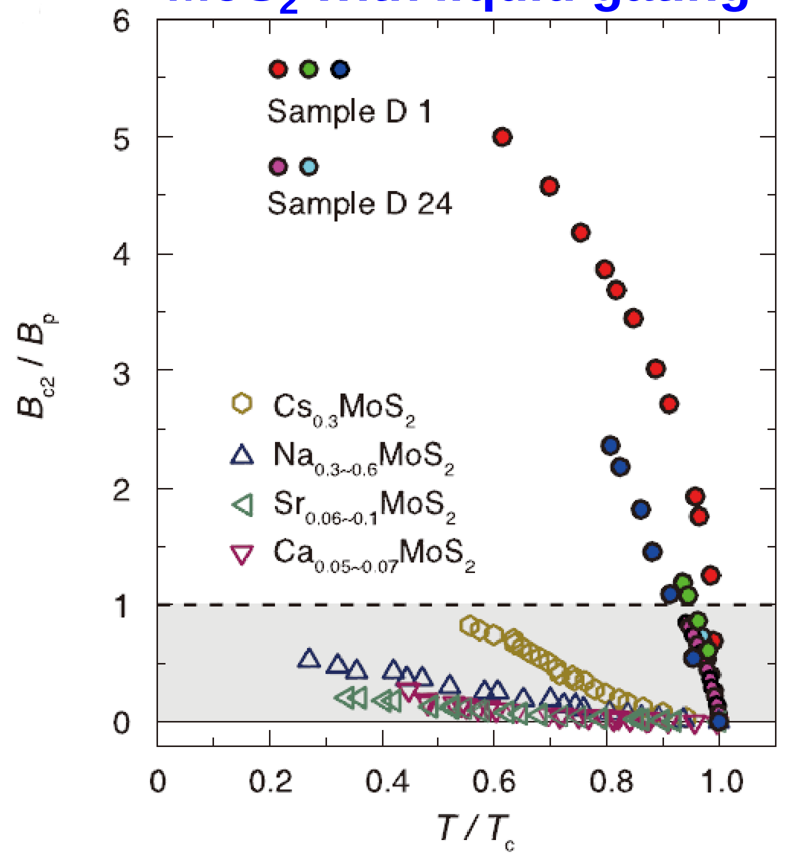
Peter Fulde, Cooper pair breaking, Chapter 11 BCS: 50 years, World scientific (2011).

Previous theories mainly consider impurity and spin orbital scattering, without considering the spin-orbital effect from the specific band structure.

Inversion asymmetric Ising superconductivity

Ising superconductivity in TMDs

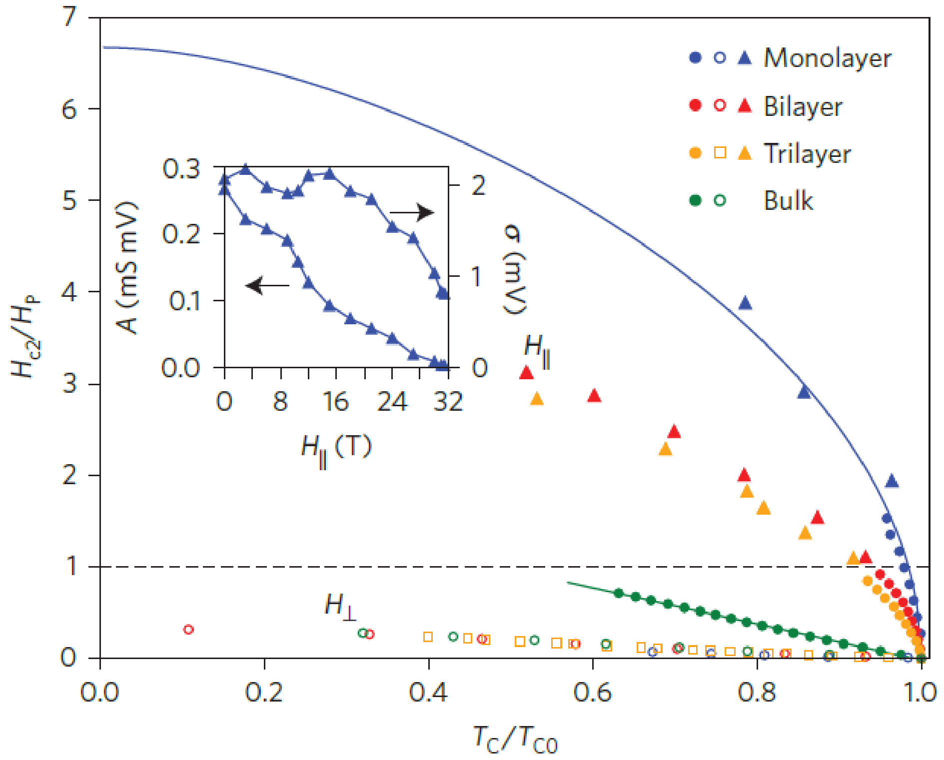
MoS_2 with liquid gating



J. M. Lu, et al., Science 350,1353(2015).

Yu Saito et al., Nat. Phys. 12, 144 (2015).

$NbSe_2$



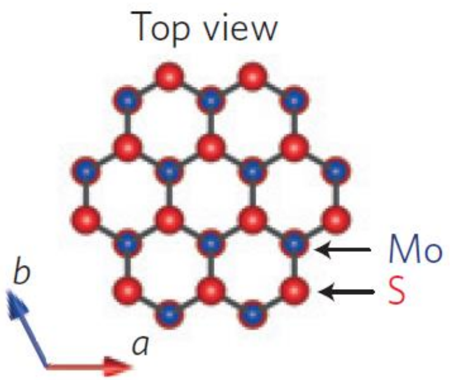
Xiaoxiang Xi, et al., Nat. Phys. 12, 139 (2015).

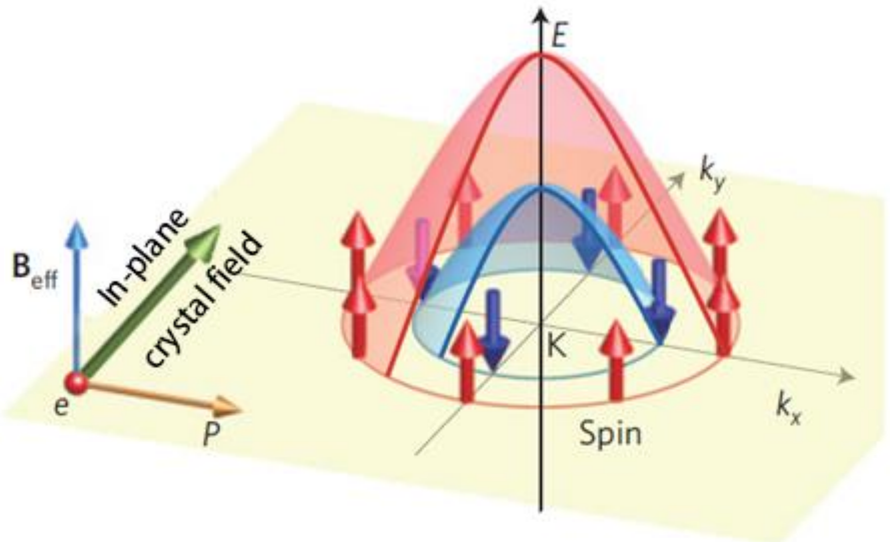
Ising superconductivity is characterized by a in-plane Bc beyond Pauli limit.

Inversion asymmetric Ising superconductivity

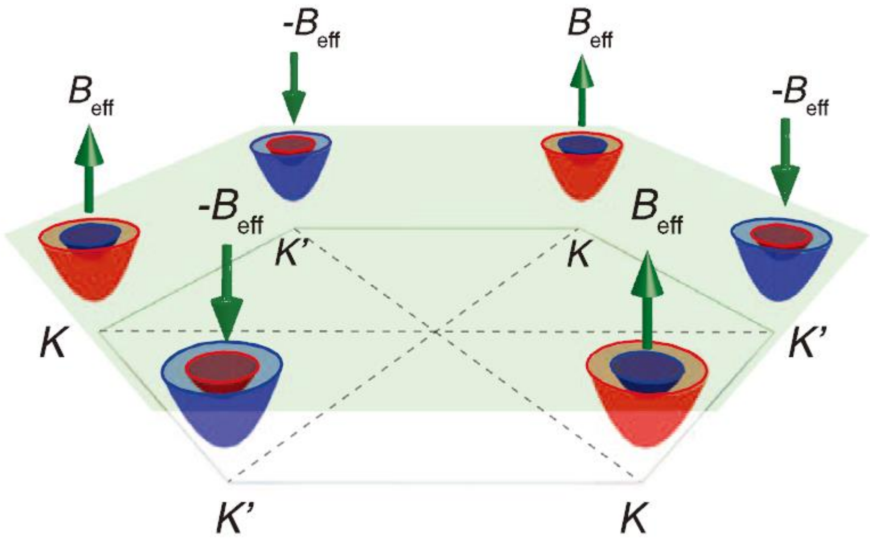
Take MoS_2 for an example

In-plane inversion symmetry breaking





Zeeman-like effective magnetic field is induced at K(K') valley of MoS_2

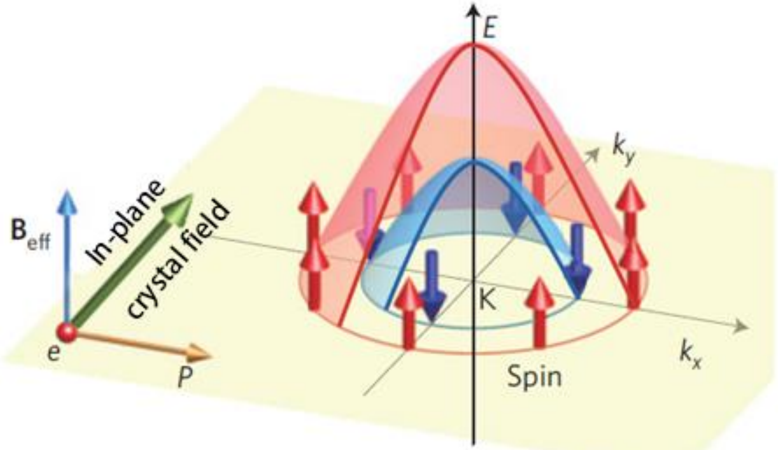


- **B_{eff} originates from intrinsic Zeeman-type spin-orbit interaction (SOI), and the spin orientation of electrons is pinned by B_{eff} in a direction normal to the film.**
- **The superconductivity is protected by this Zeeman-type SOI.**

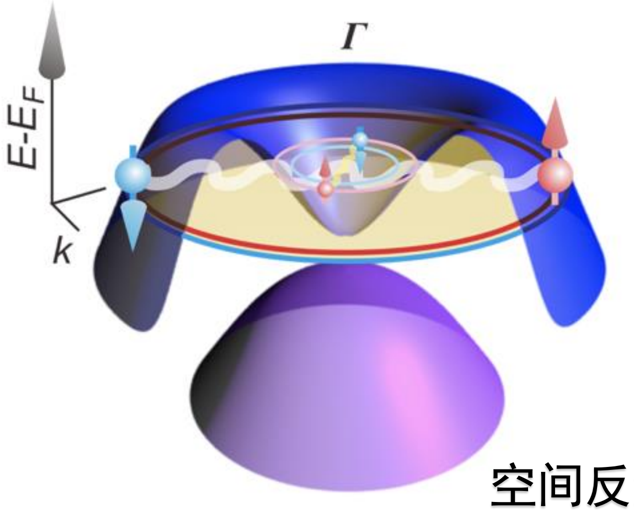
J. M. Lu, et al., Science 350,1353(2015).
Xiaoxiang Xi, et al., Nat. Phys. 12, 139 (2015).
Yu Saito et al., Nat. Phys. 12, 144 (2015).

Microscopic theory of Ising superconductivity

- 之前的伊辛超导配对机制仅适用于空间反演破缺的体系
- 在微观层面上定量处理杂质散射、能带效应、Rashba自旋轨道耦合



空间反演破缺体系



空间反演对称体系

提出新的伊辛配对机制，适用于空间反演对称体系

基于杂质格林函数和微观模型，给出定量结果

H.-C. Liu, H. Liu*, D. Zhang, X. C. Xie, arXiv:1909.08819.

Obtaining the in-plane critical field

The transition temperature T satisfies:

$$ln\left(\frac{T_c}{T}\right) = k_B T \sum_n \left[\frac{\pi}{|\omega|} - \frac{1}{2} tr\left\{\frac{F_\omega(0)}{N(0)\Delta}\right\}\right]$$

The anomalous Green's function:

$$F_\omega(\vec{r} - \vec{r}') = \iint d^3\overrightarrow{r_1} d^3\overrightarrow{r_2} G^n_{\omega,\sigma}(\vec{r} - \overrightarrow{r_1})\Delta_\omega(\overrightarrow{r_1} - \overrightarrow{r_2})G^n_{-\omega,-\sigma}(\vec{r}' - \overrightarrow{r_2})$$

The Gap function generally reads:

$$\Delta_\omega(\vec{r} - \vec{r}') = \delta^3(\vec{r} - \vec{r}')\Delta + \iint d^3\vec{r} d^3\vec{r}' \langle V(\vec{r} - \overrightarrow{r_1})F_\omega(\overrightarrow{r_1} - \overrightarrow{r_2})V(\overrightarrow{r_2} - \vec{r}')\rangle$$

The integral kernel function is introduced:

$$S_\omega(\hat{p}) \equiv \frac{k_B T}{\Delta}\int d\xi F_\omega(\vec{p})$$

$$F_\omega(0) = \int F_\omega(\vec{p})\, d^2\vec{p} \equiv \frac{\Delta}{k_B T}\int d^2\hat{p}\, S_\omega(\hat{p}) = \frac{\Delta N(0)}{k_B T} S_\omega(\hat{p})$$

H. C. Liu, H. Liu*, D. Zhang, X. C. Xie, arXiv:1909.08819.

Obtaining the in-plane critical field

Bare anomalous Green's function without scattering reads:

$$F_{\omega}^{0}(\vec{r}-\vec{r}') = \Delta \iint d^3\overrightarrow{r_1} G_{\omega,\sigma}^{n}(\vec{r}-\overrightarrow{r_1}) G_{-\omega,-\sigma}^{n}(\vec{r}'-\overrightarrow{r_1})$$

Spin-independent scattering disorder gives a finite life time $\frac{\hbar}{\tau_0} = 2\pi n_i N(0) V^2$.

$$S_{\omega}(\hat{p}) = S_{\omega}^{0}(\hat{p}) \left[1 + \frac{n_i N(0) V^2}{k_B T} S_{\omega}(\hat{p}) \right]$$

Spin-independent scattering τ_0 and spin orbital scattering τ_{so}, and $\tau^{-1} = \tau_0^{-1} + \tau_{so}^{-1}$

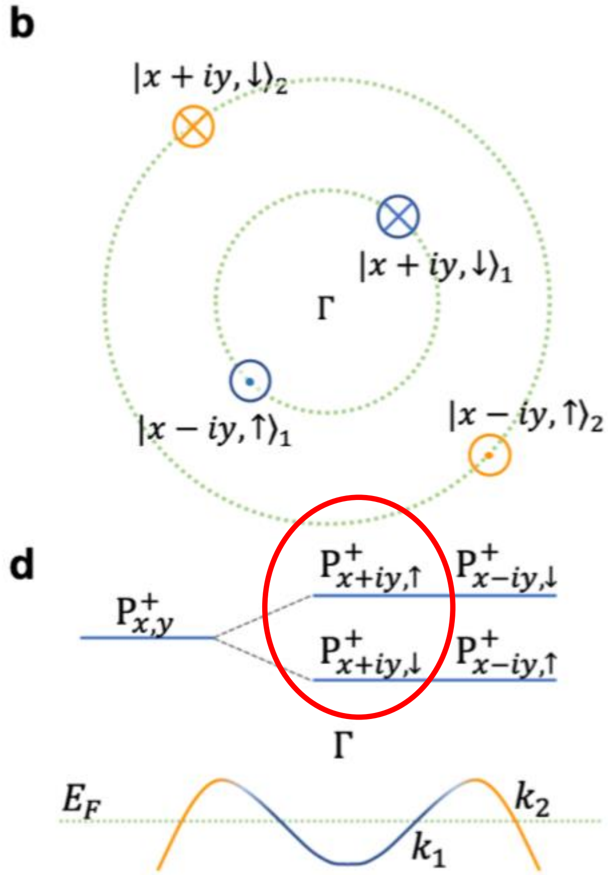
$$S_{\omega}(\hat{p}) = S_{\omega}^{0}(\hat{p}) \left\{ 1 + \frac{\tau^{-1} \cdot S_{\omega}(\hat{p})}{2\pi k_B T} - \frac{4}{3} \tau_{so}^{-1} \left[S_{\omega}^{(2)}(\hat{p}) \sigma_x I_2 + S_{\omega}^{(3)}(\hat{p}) \sigma_y \tau_z \right] / 2\pi k_B T \right\}$$

After obtaining the normal Green function $G_{\omega,\sigma}^{n}(\vec{r}-\overrightarrow{r_1})$ one can obtain the critical field And the transition temperature T satisfies: $ln\left(\frac{T_c}{T}\right) = k_B T \sum_n \left[\frac{\pi}{|\omega|} - \frac{1}{2} tr \left\{ \frac{F_{\omega}(0)}{N(0)\Delta} \right\} \right]$

H. C. Liu, H. Liu*, D. Zhang, X. C. Xie, arXiv:1909.08819.

Microscopic theory of inversion symmetric Ising superconductivity

type-II Ising superconductivity (TIs)



Band degeneracy around Gamma point.

The four band model of normal state Stanene with Rashba SOC:

$$H_{\mathrm{II}}(\boldsymbol{k}) = Ak^2 + \begin{bmatrix} H_+(\boldsymbol{k}) & -\mu_{\mathrm{B}} B\sigma_x \\ -\mu_{\mathrm{B}} B\sigma_x & H_-(\boldsymbol{k}) \end{bmatrix}$$

$$H_\pm(\boldsymbol{k}) = \begin{bmatrix} M_0 - M_1 k^2 & v(\pm k_x - ik_y) \\ v(\pm k_x + ik_y) & -M_0 + M_1 k^2 \end{bmatrix}$$

$$H_{\mathrm{R}} = -\alpha_{\mathrm{R}}(k_y \tau_x - k_x \tau_y)\sigma_x$$

In-plane critical field satisfies with Rashba SOC:

$$ln\frac{T_c}{T_{c,0}} + \frac{1}{2}(G_+ + G_-)\frac{\mu_B^2 B^2}{\beta_{so}^{*2} + \mu_B^2 B^2} = 0$$

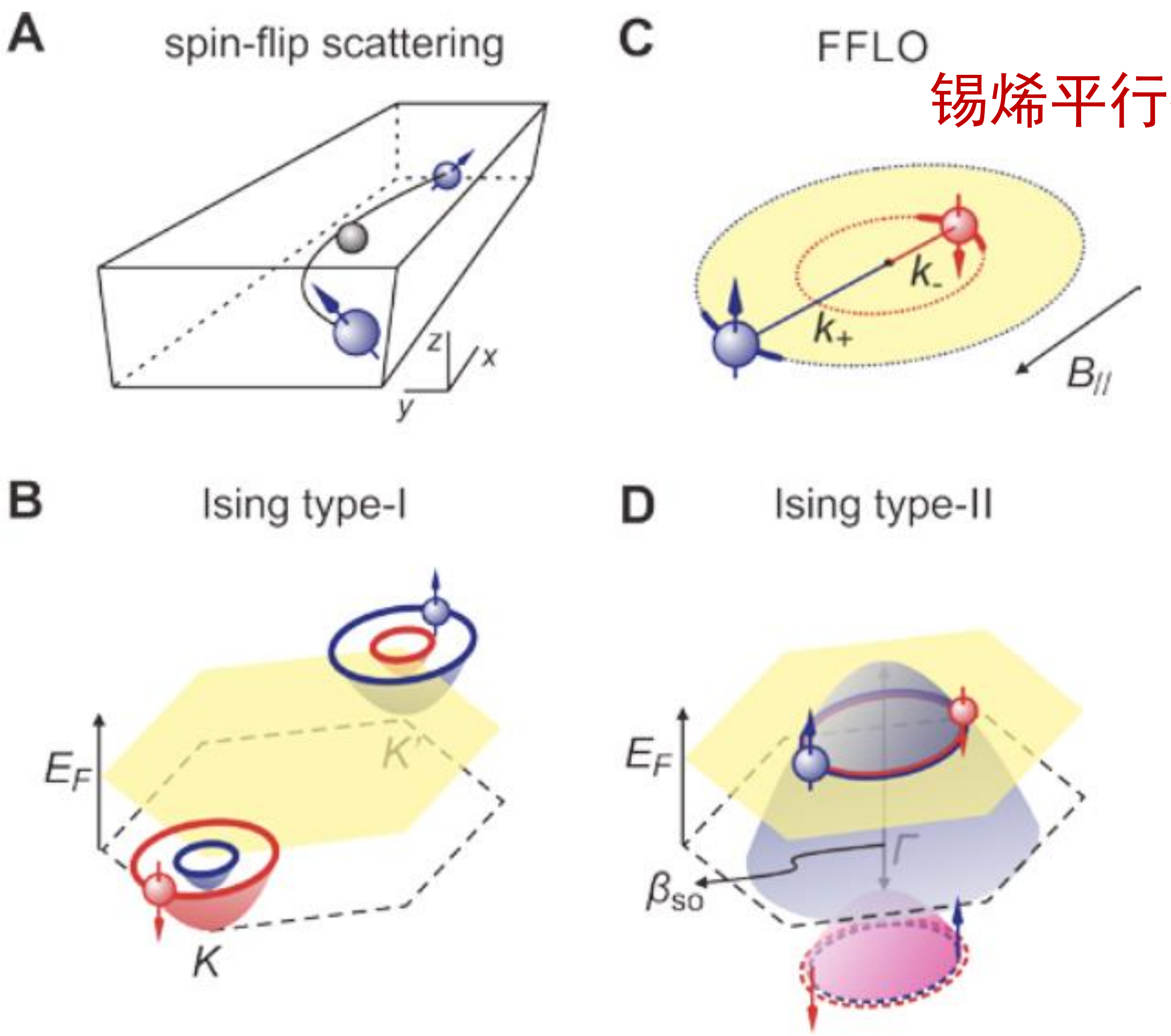
$$G_\pm = \left(1 \mp \frac{2(\alpha k_F)^2 + \beta_{so}^{*2} - \mu_B^2 B^2}{\rho_+^2 - \rho_-^2}\right) Re\left[\psi\left(\frac{1}{2} + \frac{i\rho_\pm}{2\pi k_B T_c}\right) - \psi\left(\frac{1}{2}\right)\right]$$

$$2\rho_\pm = \sqrt{(\mu_B B + \alpha k_F)^2 + (\alpha k_F)^2 + \beta_{so}^2} \pm \sqrt{(\mu_B B - \alpha k_F)^2 + (\alpha k_F)^2 + \beta_{so}^{*2}}$$

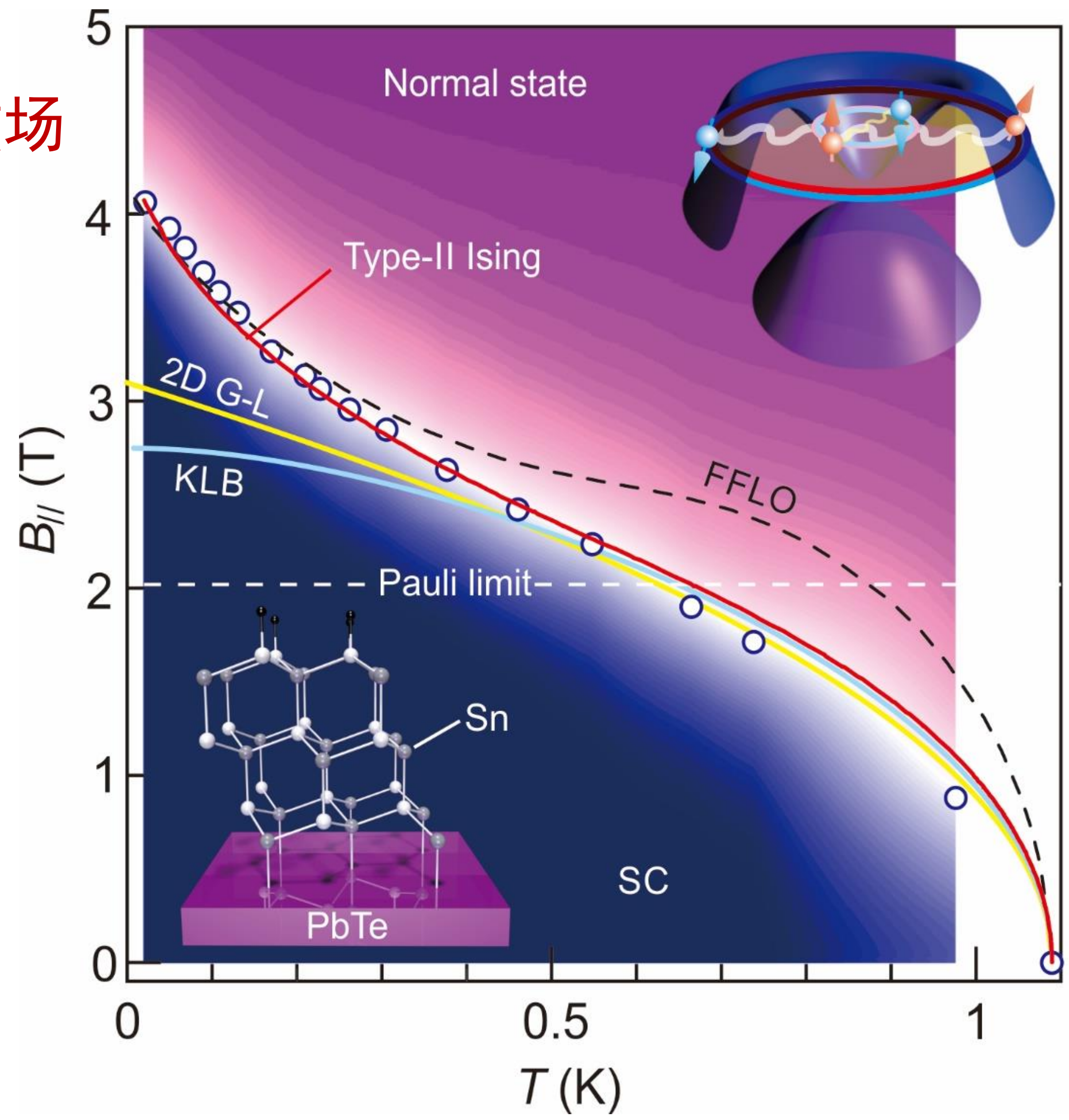
$$\beta_{so}^* = \frac{\sqrt{\left(M_0 - M_1 k_F^2\right)^2 + v^2 k_F^2}}{1 + \hbar/2\pi\tau_0 \cdot k_B T_{c,0}}$$

H. C. Liu, H. Liu*, D. Zhang, X. C. Xie, arXiv:1909.08819.

Ising superconductivity with Bc exceed the Pauli limit

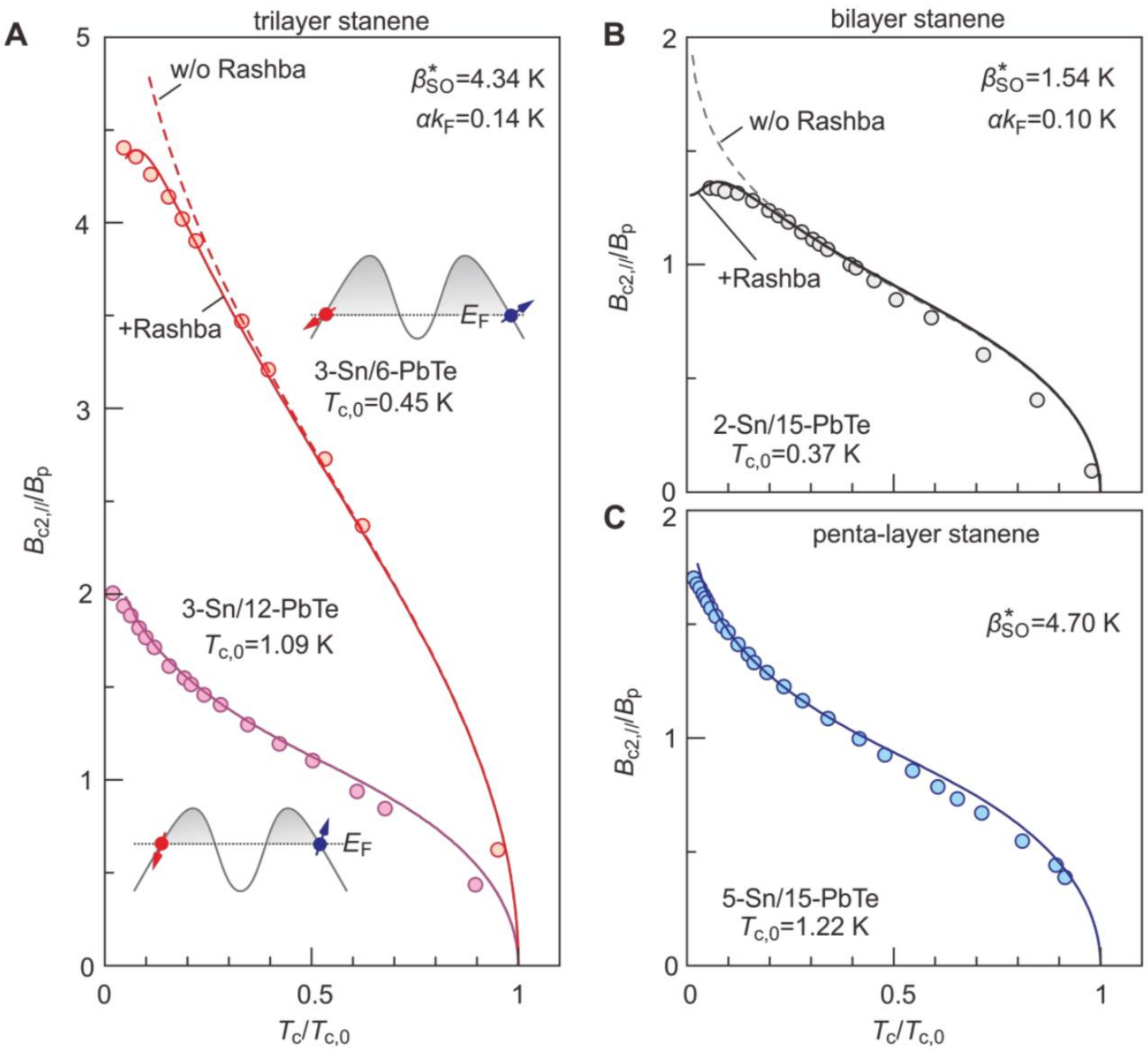


锡烯平行磁场



J. Falson, ..., H. Liu*, J. Smet*, D. Zhang*, Q. K. Xun, Science **367**, 1454(2020).

Applying to inversion symmetric Ising superconductivity in stanene films



锡烯平行磁场

1, Largely exceeds the Pauli-limit.

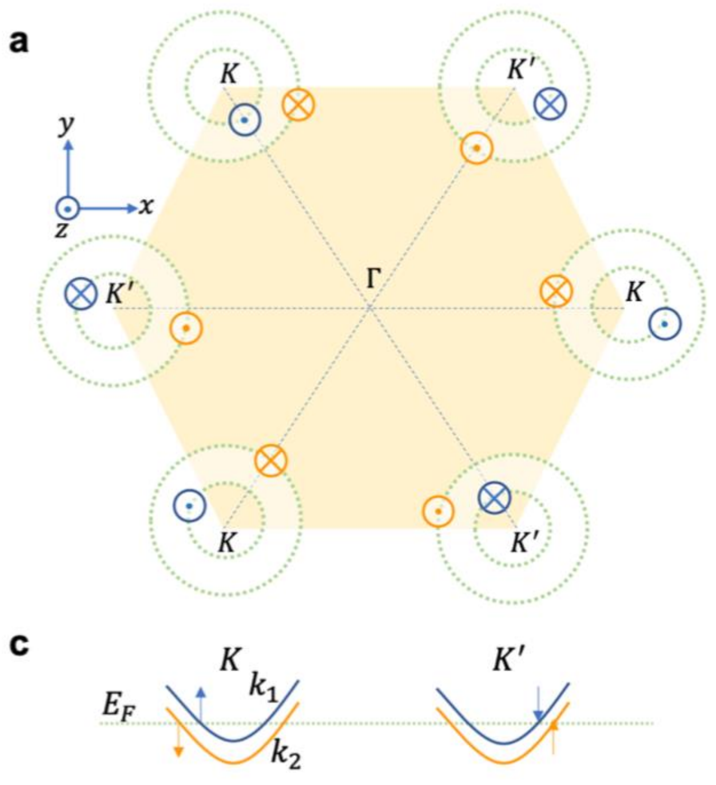
2, Remarkable up-turn near zero temperature.

3, type-II Ising superconductivity in stanene films with spin-degeneracy.

Type-II Ising superconductivity with full features.

J. Falson, ..., H. Liu*, J. Smet*, D. Zhang*, Q. K. Xun, Science **367**, 1454(2020).

Microscopic theory of inversion asymmetric Ising superconductivity



Band splitting around K and K' points.

Ising superconductivity In Pb films with Rashba SOI

$$H_{\vec{p},K} = \frac{p^2}{2m} - \beta_{SO}\sigma_z\tau_z - \mu_B B\sigma_x - \alpha_R(\sigma_x k_y - \sigma_y k_x)$$

$$H_{\vec{p},K'} = \frac{p^2}{2m} - \beta_{SO}\sigma_z\tau_z - \mu_B B\sigma_x + \alpha_R(\sigma_x k_y - \sigma_y k_x)$$

Bc satisfies the relation:

$$ln\left(\frac{T_c}{T}\right) = \frac{1}{2}\left[1 - \frac{2(\widetilde{\alpha_R k_F})^2 + \widetilde{\beta_{SO}}^2 - \mu_B^2 B^2}{\rho_+^2 - \rho_-^2}\right] Re\left[\psi\left(\frac{1}{2} + \frac{i\rho_+}{2\pi k_B T}\right) - \psi\left(\frac{1}{2}\right)\right]$$
$$+\frac{1}{2}\left[1 + \frac{2(\widetilde{\alpha_R k_F})^2 + \widetilde{\beta_{SO}}^2 - \mu_B^2 B^2}{\rho_+^2 - \rho_-^2}\right] Re\left[\psi\left(\frac{1}{2} + \frac{i\rho_-}{2\pi k_B T}\right) - \psi\left(\frac{1}{2}\right)\right]$$

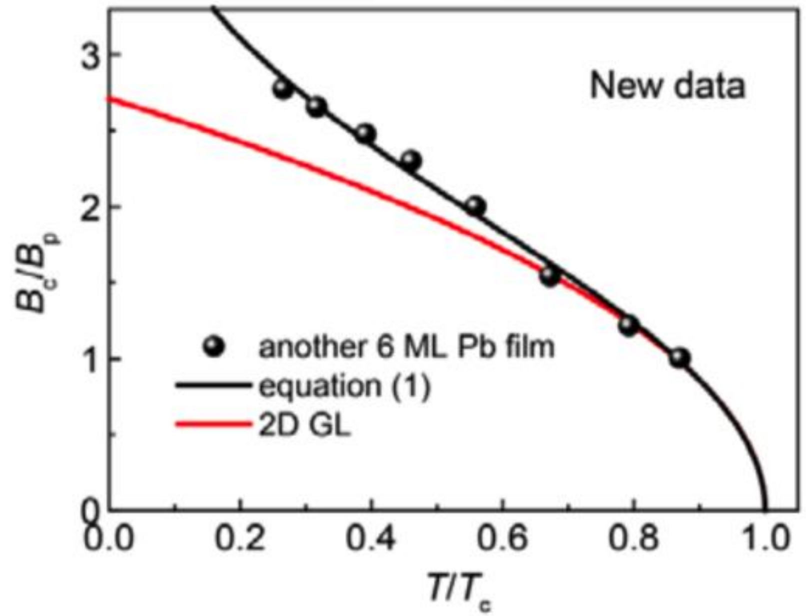
$$2\rho_\pm \equiv \sqrt{(\mu_B B + \widetilde{\alpha_R k_F})^2 + (\widetilde{\alpha_R k_F})^2 + \widetilde{\beta_{SO}}^2} \pm \sqrt{(\mu_B B - \widetilde{\alpha_R k_F})^2 + (\widetilde{\alpha_R k_F})^2 + \widetilde{\beta_{SO}}^2}$$

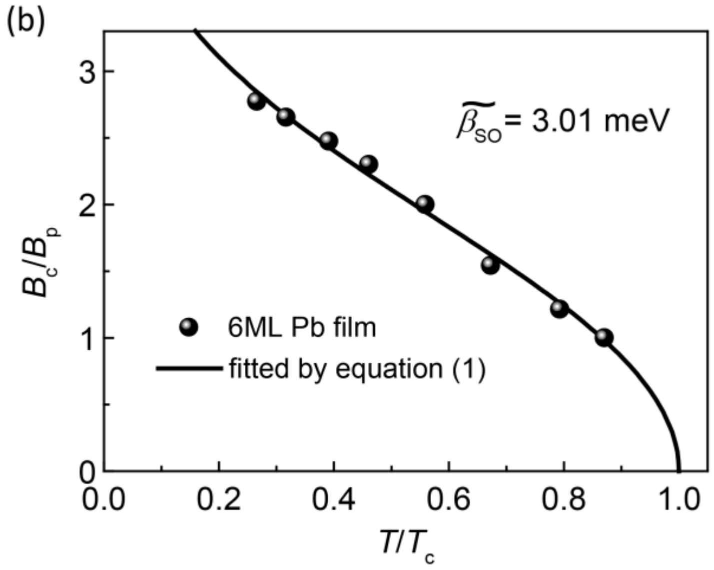
$$\widetilde{\beta_{SO}} = \beta_{SO}/(1 + \frac{\hbar}{2\pi k_B T_c \tau_0}) \qquad \widetilde{\alpha_R k_F} \equiv \frac{\alpha_R k_F}{\sqrt{2}\left(1 + \frac{\hbar}{2\pi k_B T \tau_0}\right)}$$

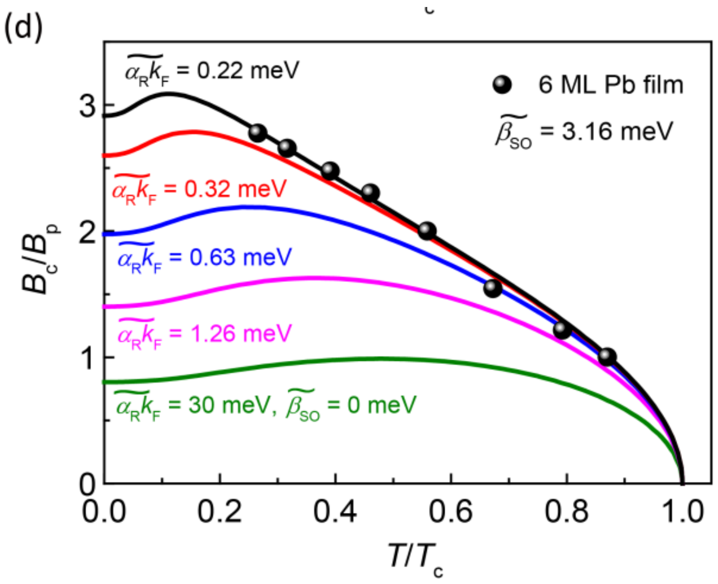
Y. Liu, ..., H. Liu*, et al., J. Wang*, Physical Review X 8, 021002 (2018).

Applying to inversion asymmetric Ising superconductivity in Pb films

铅薄膜平行磁场







Parallel field

6 ML Pb film

The microscopic theory for type-I superconductivity:

Considering both Zeeman-type SOI $\widetilde{\beta_{SO}}$ and Rashba-type SOI $\widetilde{\alpha_R k_F}$

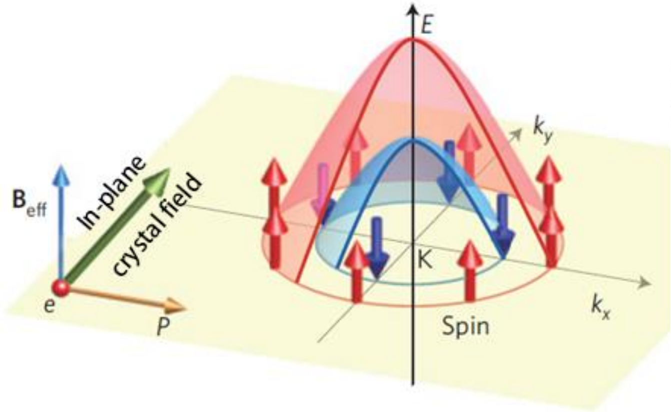
$$ln\left(\frac{T_c}{T}\right) = \frac{1}{2}\left[1 - \frac{2\left(\widetilde{\alpha_R k_F}\right)^2 + \widetilde{\beta_{SO}}^2 - \mu_B^2 B^2}{\rho_+^2 - \rho_-^2}\right] Re\left[\psi\left(\frac{1}{2} + \frac{i\rho_+}{2\pi k_B T}\right) - \psi\left(\frac{1}{2}\right)\right]$$
$$+\frac{1}{2}\left[1 + \frac{2\left(\widetilde{\alpha_R k_F}\right)^2 + \widetilde{\beta_{SO}}^2 - \mu_B^2 B^2}{\rho_+^2 - \rho_-^2}\right] Re\left[\psi\left(\frac{1}{2} + \frac{i\rho_-}{2\pi k_B T}\right) - \psi\left(\frac{1}{2}\right)\right]$$

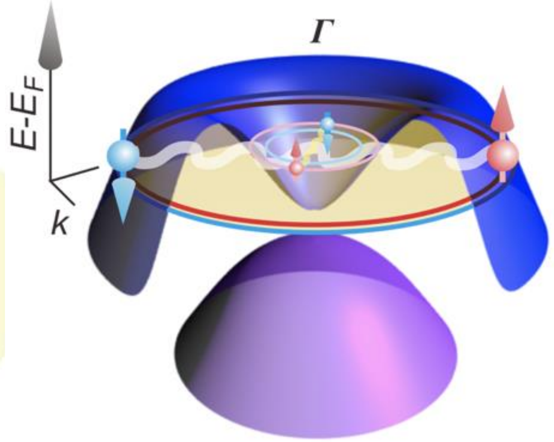
$$\widetilde{\alpha_R k_F} \equiv \frac{\alpha_R k_F}{\sqrt{2}\left(1 + \frac{\hbar}{2\pi k_B T \tau_0}\right)}$$

$$\widetilde{\beta_{SO}} = \beta_{SO}/(1 + \frac{\hbar}{2\pi k_B T_c \tau_0})$$

Summary for microscopic theory of Ising superconductivity

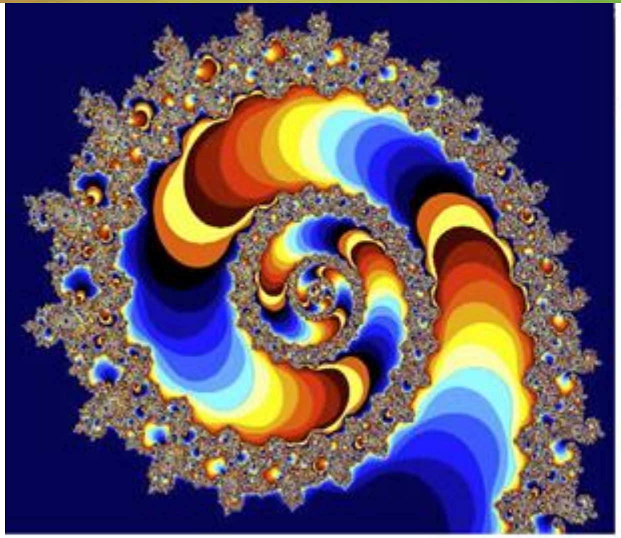
Our microscopic theory can give quantitatively explanation for measurements in 2D superconducting stanene and Pb ultrathin films, which are largely deviated from the prediction of Ginzburg-Landau theory and previous microscopic theories.



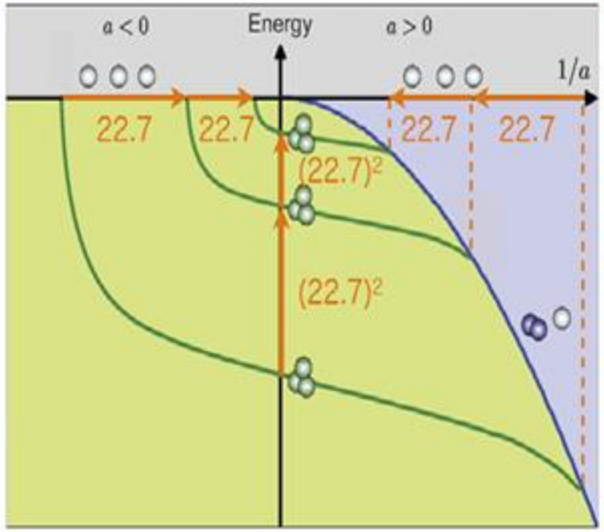


Deviation from the power law behavior:
Manifestation of discrete scale invariance

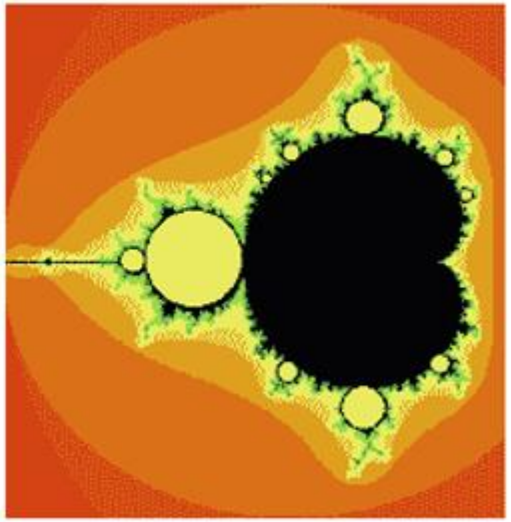
Introduction to Discrete Scale Invariance



Fibonacci (Spirals) Fractals



Efimov spectrum



Mandelbrot set

Benoit B. Mandelbrot
1924-2010

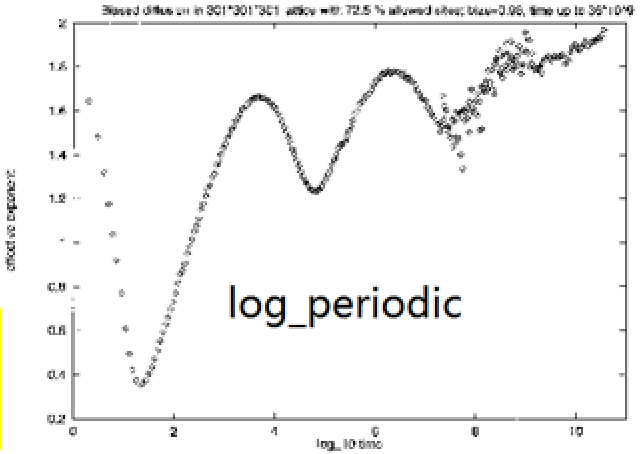
$$\mu F(x) = F[\phi(x)]$$

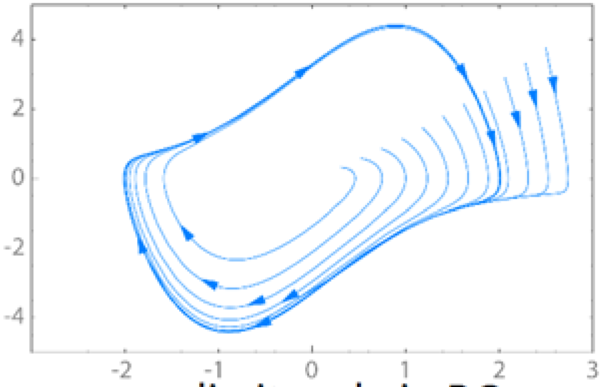
$$\phi(x) = \lambda x + \ldots$$

$$\omega = \ln \mu / \ln \lambda$$

$$F_0(x) = x^{\omega}$$

$$F(x) = x^{\omega}\left[1 + \sin\left(2\pi \frac{\ln x}{\ln \lambda} + \alpha\right)\right]$$





D. Sornette, Physics Reports 297 239 (1998).
E. Braaten, H.-W. Hammer, Physics Reports 428 259 (2006).

Bosonic Efimov trimers with resonant scattering

Volume 33B, number 8 PHYSICS LETTERS 21 December 1970

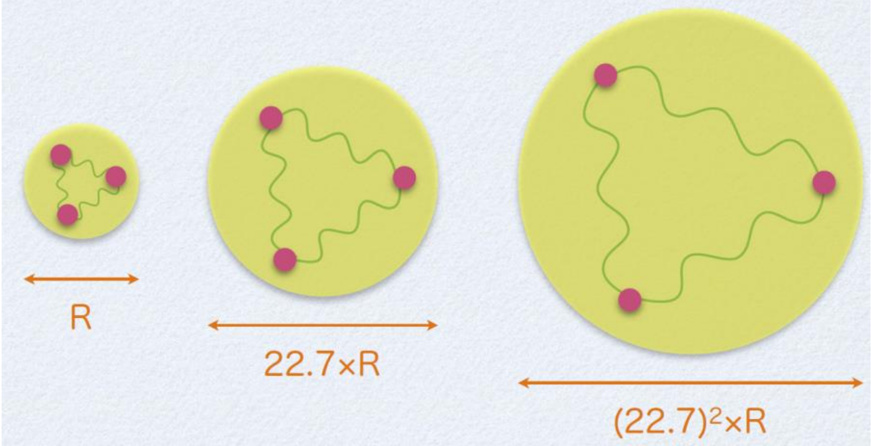
ENERGY LEVELS ARISING FROM RESONANT TWO-BODY FORCES IN A THREE-BODY SYSTEM

V. EFIMOV
A.F.Ioffe Physico-Technical Institute, Leningrad, USSR

Received 20 October 1970

Vitaly Efimov

When 2 bosons interact with infinite scattering length a, 3 bosons always form a geometric series of bound states.



$$e^{\pi/s_0} \approx 22.7$$
with $s_0 \approx 1.006$

Discrete Scale Invariance

Solve the differential equation with Efimov attraction

Differential equation for radical part (with real positive s_0):

$$\left[-\frac{d^2}{dR^2} - \frac{s_0^2 + 1/4}{R^2} - \frac{M_e E_T}{\hbar^2}\right] f(R) = 0$$ [Landau & Lifshitz QM]

Definitions and differential equation for auxiliary function:

$$E_T = -\frac{\hbar^2 k_T^2}{M_e} \qquad f(R) = R^{\frac{1}{2}} g(R) \qquad \tilde{R} = k_T R$$

$$g''(\tilde{R}) + \frac{1}{\tilde{R}} g'(\tilde{R}) + \left(-1 + \frac{s_0^2}{\tilde{R}^2}\right) g(\tilde{R}) = 0$$

E=Et Solution:

$$f(R) = R^{\frac{1}{2}} K_{is_0}(k_T R)$$

$$f(R) \sim R^{\frac{1}{2}} \sin[s_0 \ln(k_T R) + \alpha]$$

E≈0 Solution:

$$f_1(R) \approx A R^{\frac{1}{2}} \sin[s_0 \ln k_* R + \alpha]$$

Separate and match.

Continuous condition of $R\frac{f'(R)}{f(R)}$ at boundary.

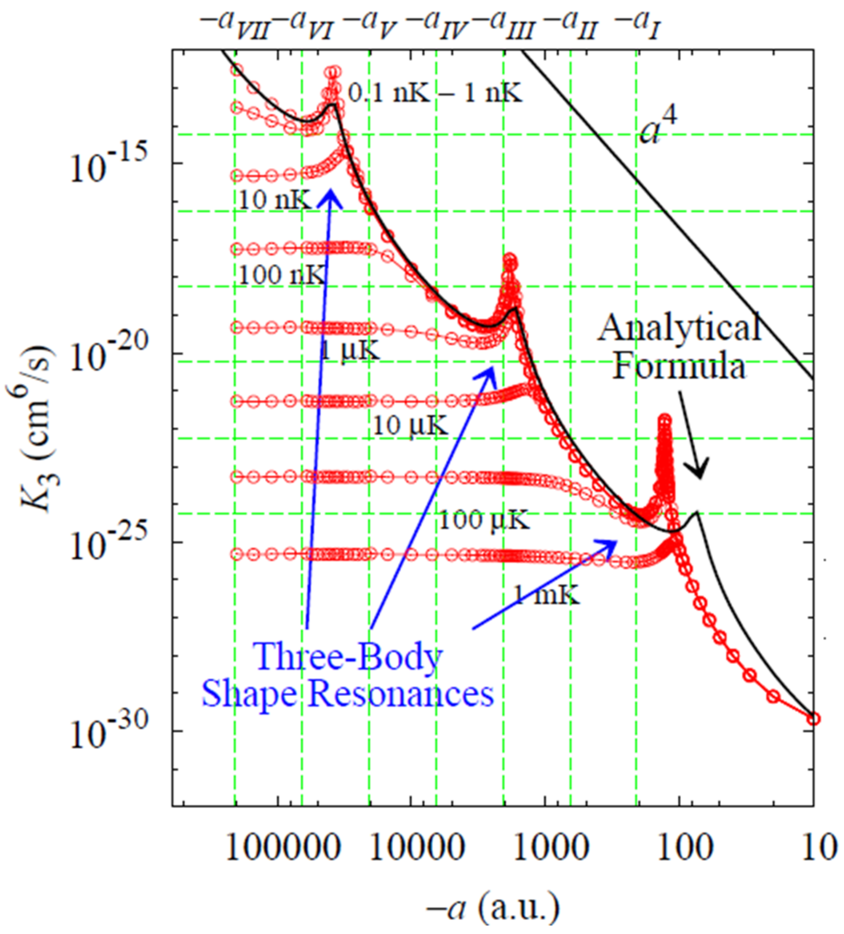
Discrete Scale Invariance

$$k_{T,n} = \left(e^{-\pi/s_0}\right)^{n-n_0} e^{-\alpha/s_0} k_*$$

$$E_{B,n} = \left(e^{-2\pi/s_0}\right)^{n-n_0} \frac{\hbar^2 \tilde{k}_*^2}{M_e}$$

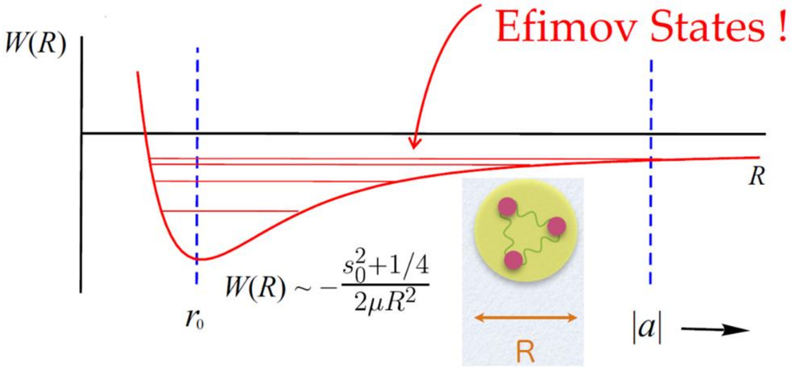
$$\langle R_n^2 \rangle \propto k_{T,n}^{-2} \qquad \frac{R_{n+1}}{R_n} = e^{\pi/s_0}$$

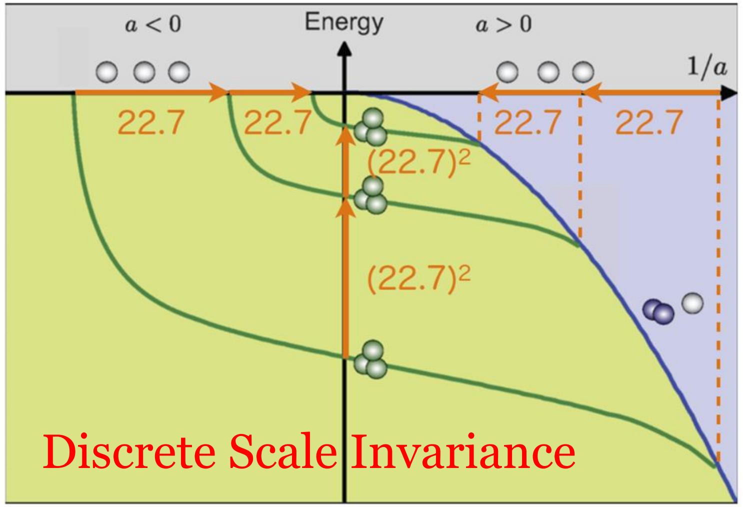
Efimov trimer of identical bosons (near resonance)



Grimm' Group, Nature 440, 315 (2006);
Dörner's Group, Science 348, 551 (2015);
Weidemüller's Group, PRL 112, 250404 (2014);
Chin's Group, PRL 113, 240402 (2014).

1/R^2 attraction is a requisite.





E. Braaten, H.-W. Hammer, Physics Reports 428 259 (2006).

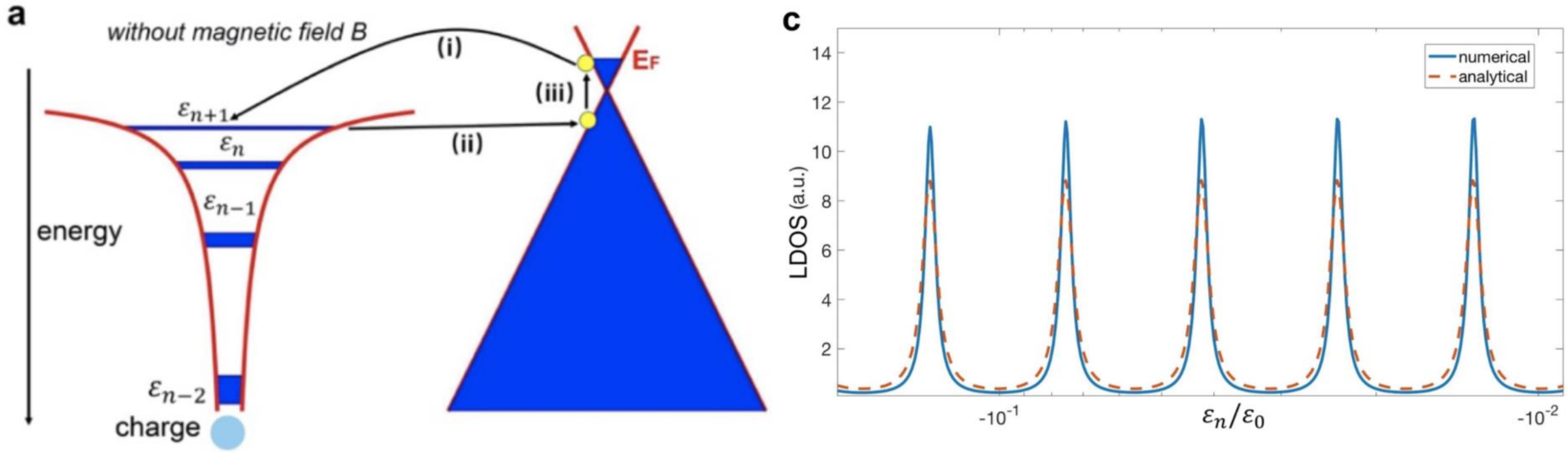
Discrete Scale Invariance in topological semimetal

Weyl 方程 + 库仑势:

$$\begin{bmatrix} 0 & \hbar v_F \vec{\sigma}\cdot\vec{k} \\ \hbar v_F \vec{\sigma}\cdot\vec{k} & 0 \end{bmatrix}\Psi = \left[E + \frac{\hbar v_F \alpha}{R}\right]\Psi$$

标度不变 + 离散能谱 =
离散标度不变

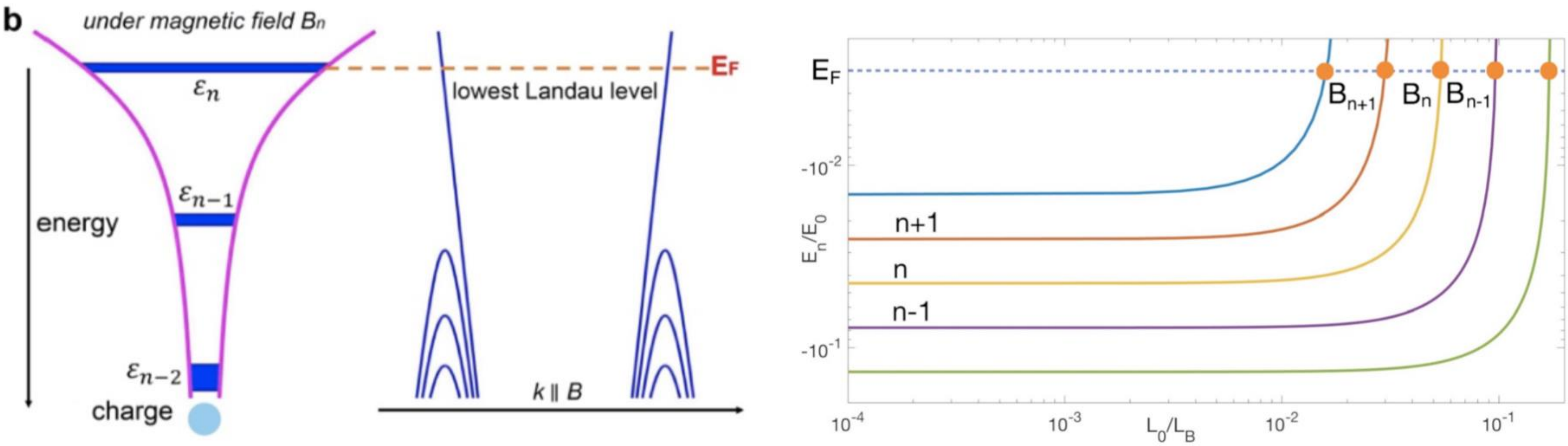
狄拉克半金属ZrTe5考虑库仑杂质势



准束缚态能级呈现等比数列。

H. Liu, *et al.*, arXiv: 1807.02459.
H.-C. Liu, H. Liu*, *et al.*, PRB **100**, 195140 (2019).

Discrete Scale Invariance in topological semimetal



准束缚态能级在磁场下演化，在特定磁场值时经过费米面。
这些磁场取值近似满足等比数列。

H. Liu, *et al.*, arXiv: 1807.02459.
H.-C. Liu, H. Liu*, *et al.*, PRB **100**, 195140 (2019).

Quasi-bound states and magneto-conductivity

$$\sigma_{xx}(\varepsilon_F) = \frac{4e^2}{h} l_B^2 \left(N_{AI} + N_{EB} \frac{t^2}{8\pi \cdot \varepsilon_* \cdot \Gamma(B)} \sum_n \frac{\Gamma(B)^2}{\left(\varepsilon_F - \varepsilon_n(B)\right)^2 + \Gamma(B)^2} \right)$$

$$\Gamma(B) = t^2 \cdot \rho(B, \varepsilon_F) = \eta^* \cdot \sqrt{B}$$

The conductivity contains two terms:

1, the first term is short-range impurity scattering (gives linear-B MR),

2, the second term is resonant scattering between mobile carrier and the quasi-bound state, which leads to logB MR oscillations.

$$\sigma_{xy} = \frac{2\pi e^2}{h} l_B^2 \cdot N$$

$$\rho_{xx}(\varepsilon) \approx \frac{\sigma_{xx}(\varepsilon)}{\sigma_{xy}^2}$$

N_{AI} short-range impurity density

N_{EB} quasi-bound states density

N mobile carrier density

Quasi-bound states and magneto-conductivity

$$\sigma_{xx}(\varepsilon_F) = \frac{4e^2}{h} l_B^2 \left(N_{AI} + N_{EB} \frac{t^2}{8\pi \cdot \varepsilon_* \cdot \Gamma(B)} \sum_n \frac{\Gamma(B)^2}{\left(\varepsilon_F - \varepsilon_n(B)\right)^2 + \Gamma(B)^2} \right)$$

$$\Gamma(B) = t^2 \cdot \rho(B, \varepsilon_F) = \eta^* \cdot \sqrt{B}$$

$$\sigma_{xx}(\varepsilon_F) = \frac{4e^2}{h} l_B^2 \left(N_{AI} + N_{EB} \frac{t^2}{8\pi \cdot \varepsilon_* \cdot \Gamma(B)} \frac{\eta^2}{sin^2\left(\frac{s_0}{2} ln\left(\frac{B}{B_0}\right)\right) + \eta^2} \right)$$

DSI

Parameterization: numerical justification will be given in the following slides.

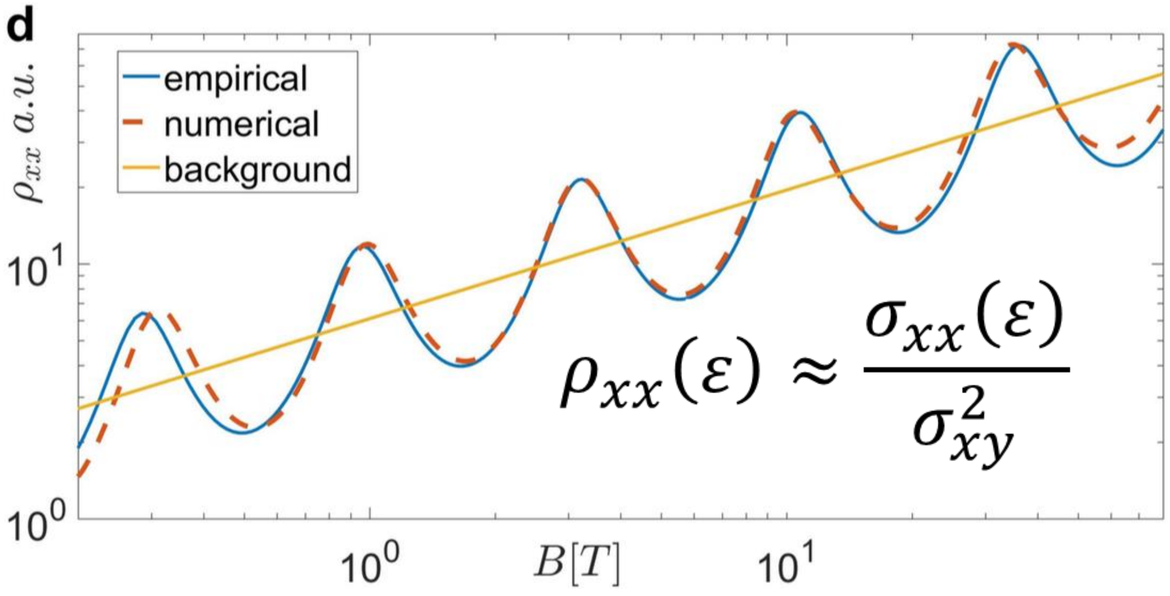
Application to the experimental results in ZrTe5

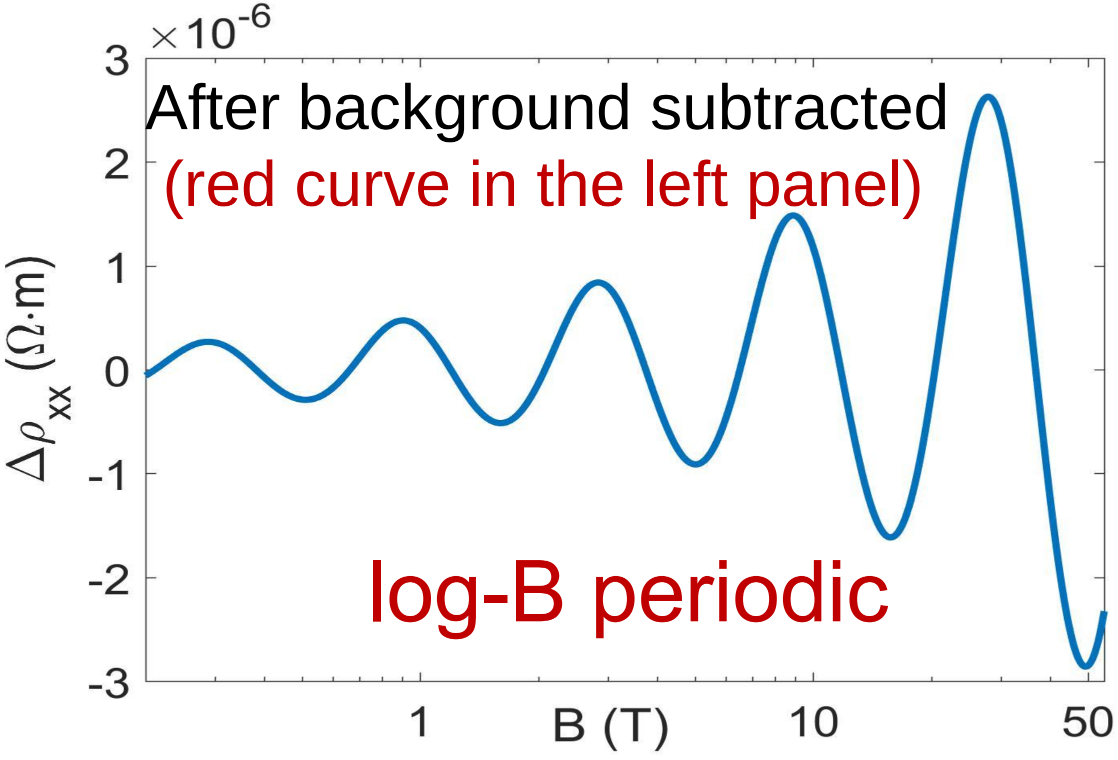
$v_F \approx 4.0 \times 10^5\ m/s$ for the Dirac bands in $ZrTe_5$ [from ARPES experiments: Nature Physics 12, 550 (2016)]

$Z\alpha \approx 5.5$ and $s_0 = \sqrt{(Z\alpha)^2 - 1} \approx 5.4$

$$\ln\frac{B_{n+1}}{B_n} \in \left(-\frac{2\pi}{s_0} - \frac{1}{s_0}, -\frac{2\pi}{s_0} + \frac{1}{s_0}\right)$$

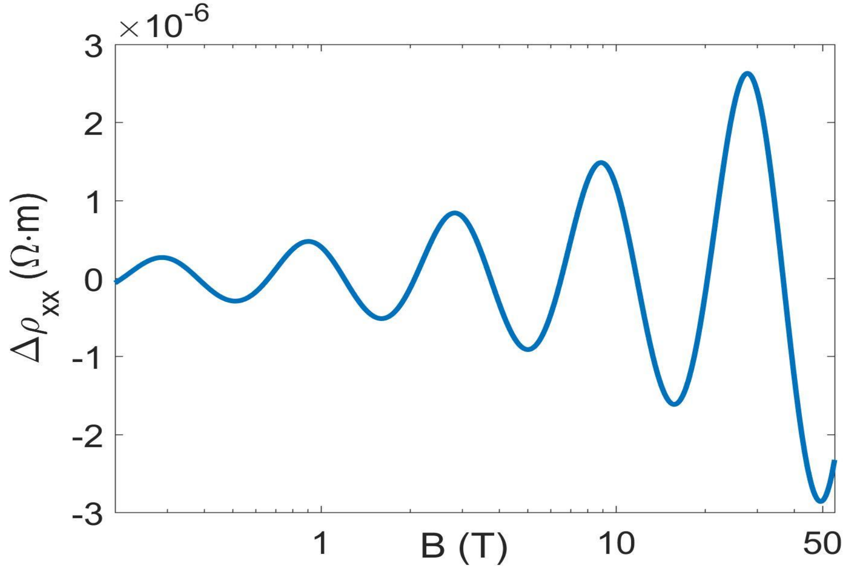
Approximate discrete scale invariance: $B_n/B_{n+1} \in (2.76, 4.06)$ Theoretical ratio

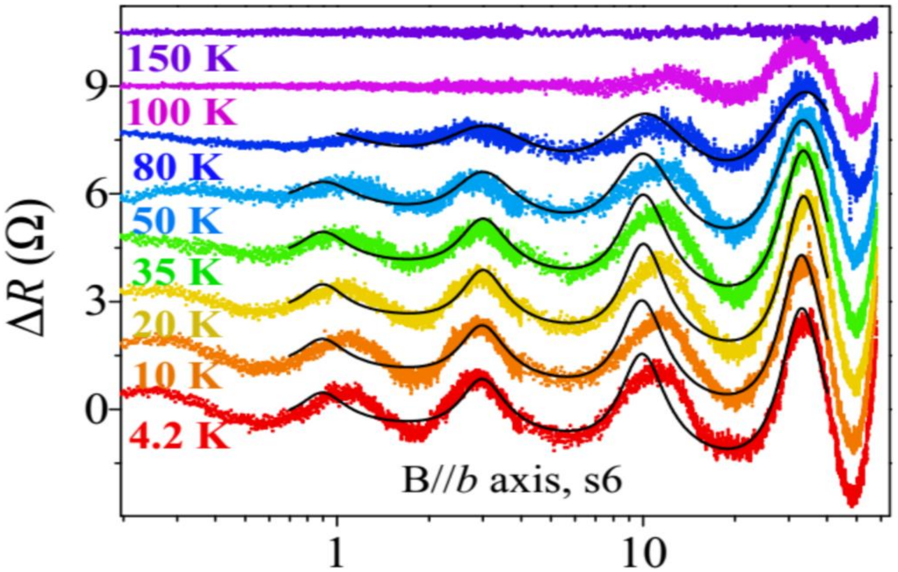




The whole curve shows DSI, not just the periods.

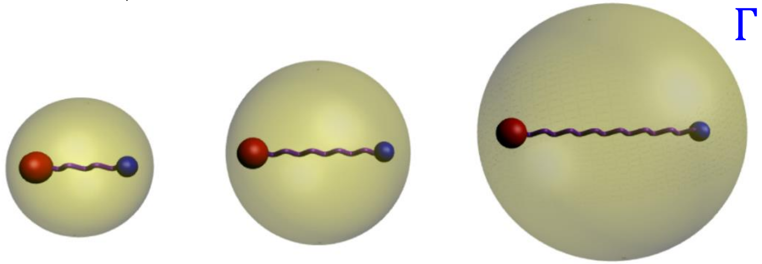
Application to the experimental results in ZrTe5





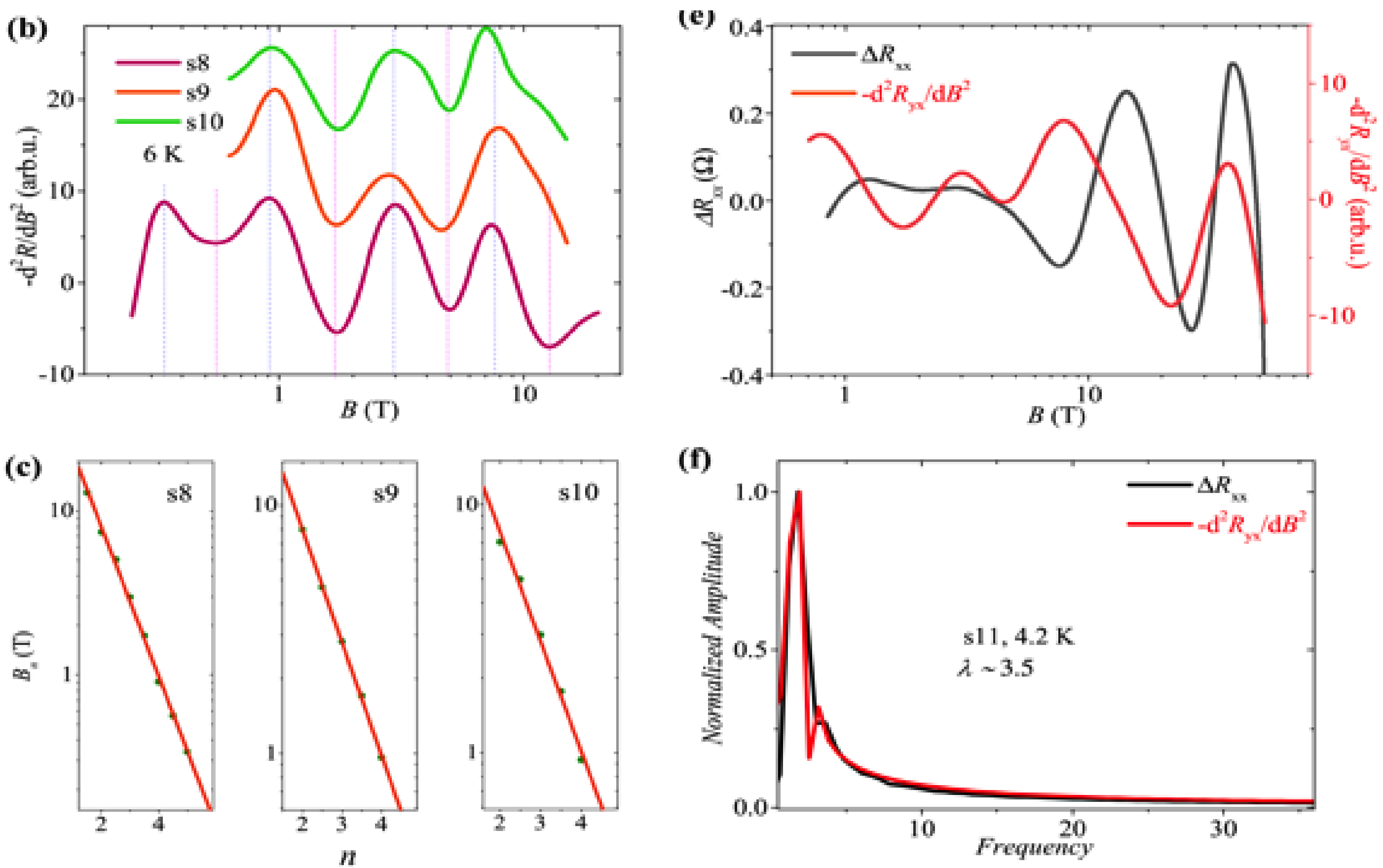
$$\sigma_{xx}(\varepsilon_F) = \frac{4e^2}{h} l_B^2 \left(N_{AI} + N_{EB} \frac{t^2}{8\pi \cdot \varepsilon_* \cdot \Gamma(B)} \sum_n \frac{\Gamma(B)^2}{\left(\varepsilon_F - \varepsilon_n(B)\right)^2 + \Gamma(B)^2} \right)$$

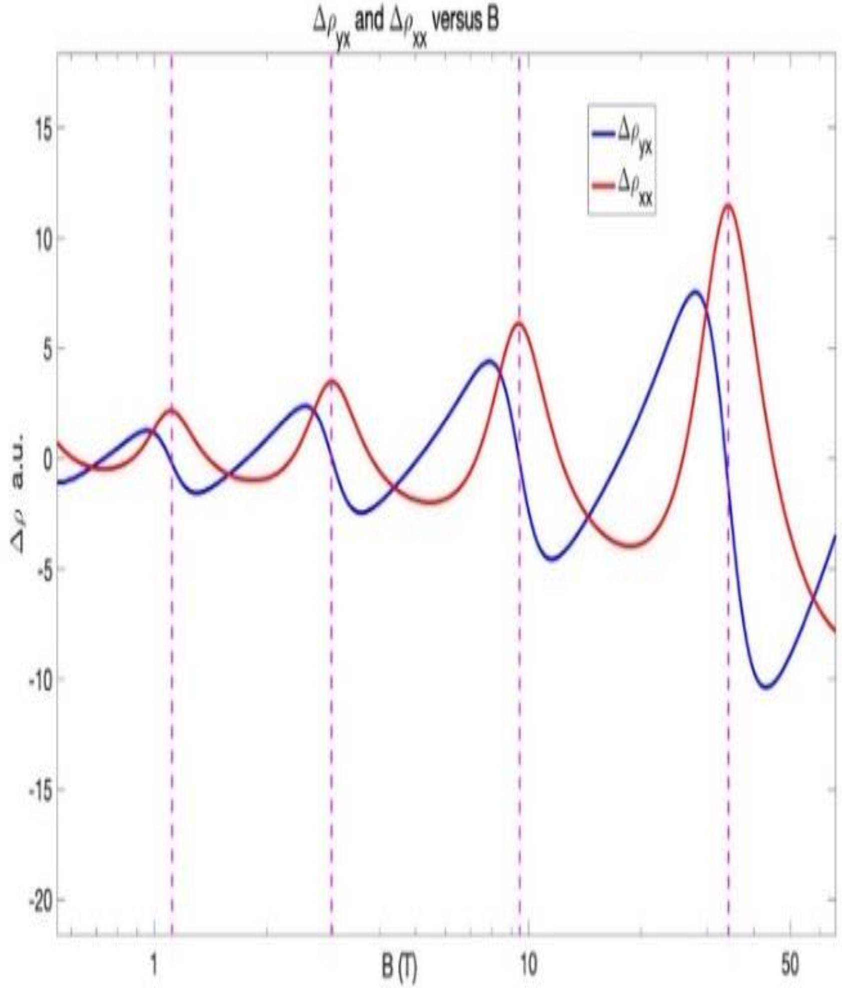
$$\Gamma(B) = \eta^* \cdot \sqrt{B}$$



H Wang#, H. Liu#, *et al.*, Science Advances **4**, eaau5096 (2018).
H. Liu, *et al.*, arXiv: 1807.02459.
H.-C. Liu, H. Liu*, *et al.*, PRB **100**, 195140 (2019).

DSI in longitudinal magneto-resistivity and Hall trace in HfTe5

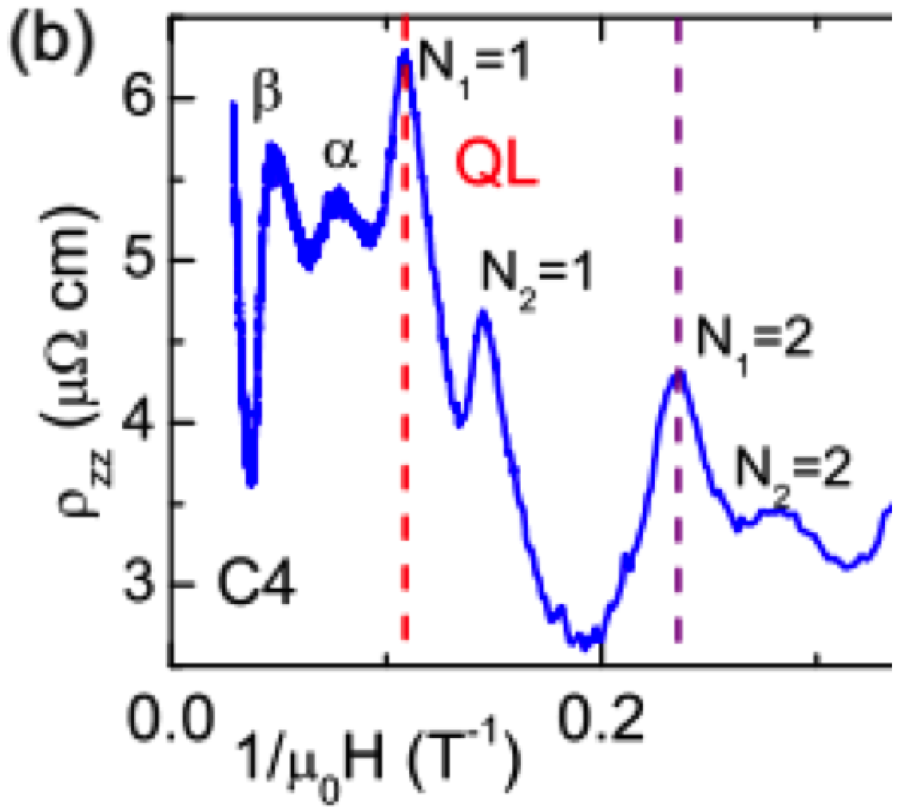




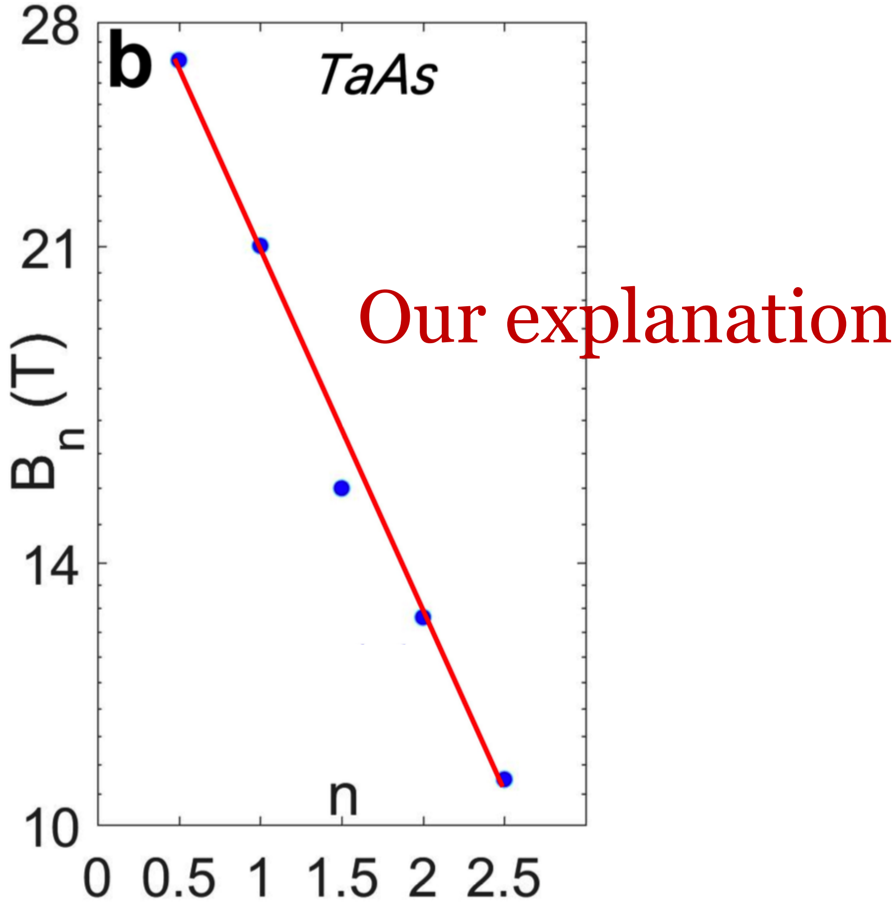
$$\sigma_{xx} = \frac{4e^2}{h} l_B^2 \left(n_s + n_C \frac{t^2}{8\pi \cdot \hbar v_F l_*^{-1} \cdot \Gamma(B)} \frac{\eta^2}{\sin^2\left(\frac{S_0}{2} \ln\left(\frac{B}{B_0}\right)\right) + \eta^2} \right)$$

Log-B periodic magneto-oscillation in TaAs

quantum oscillations beyond the quantum limit

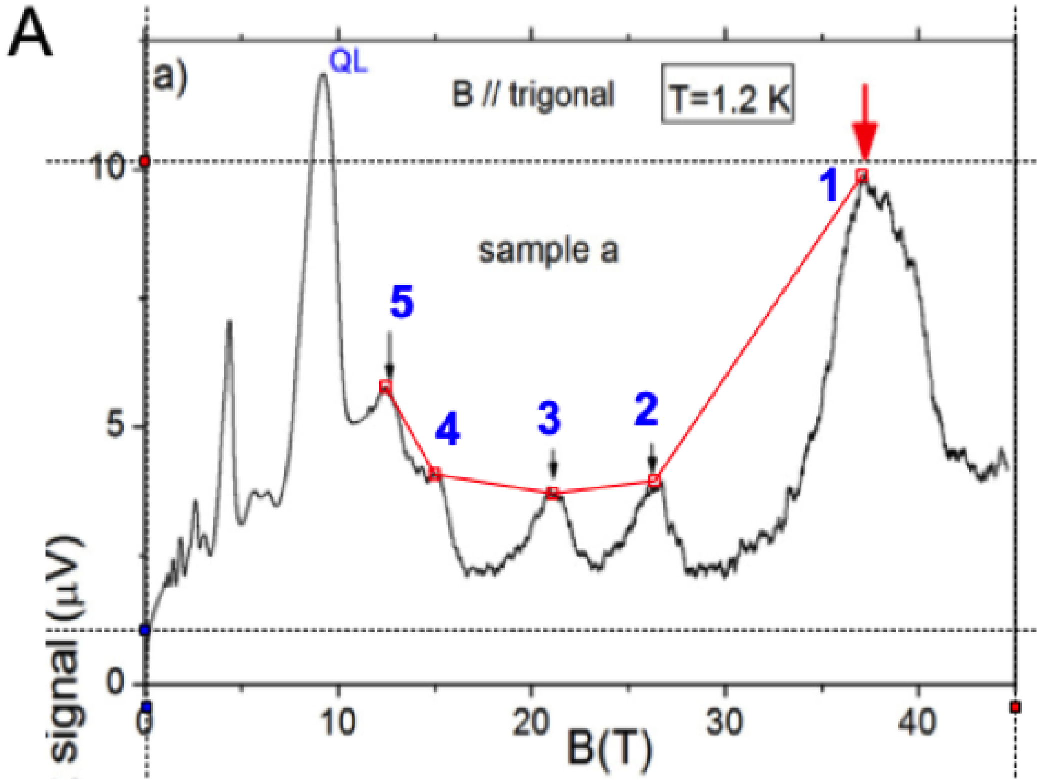


Peak and valley values of TaAs



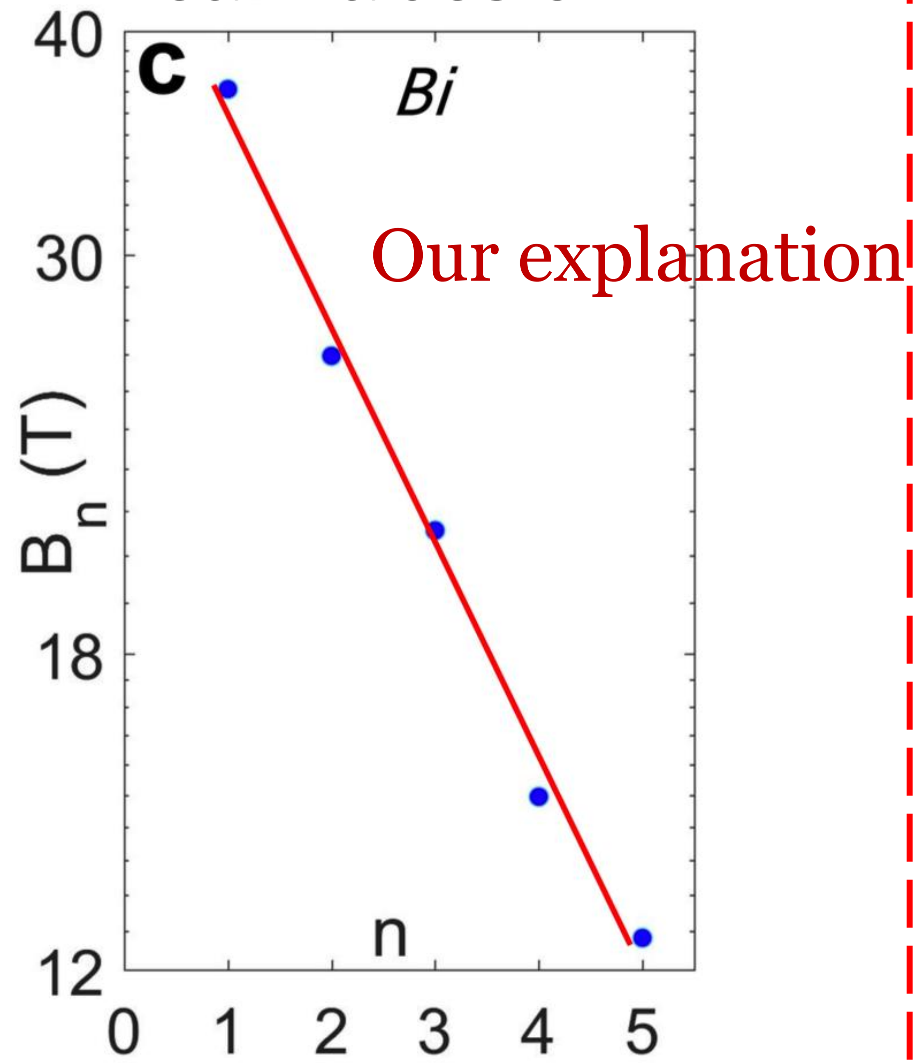
Log-periodic oscillations in TaAs magnetoresistance (B > 10T)

Log-B periodic magneto-oscillation of thermopower in Bi



K. Behnia's group, Science 317, 1729-1731 (2007).
K. Behnia's group, New J. Phys. 11, 113012 (2009).

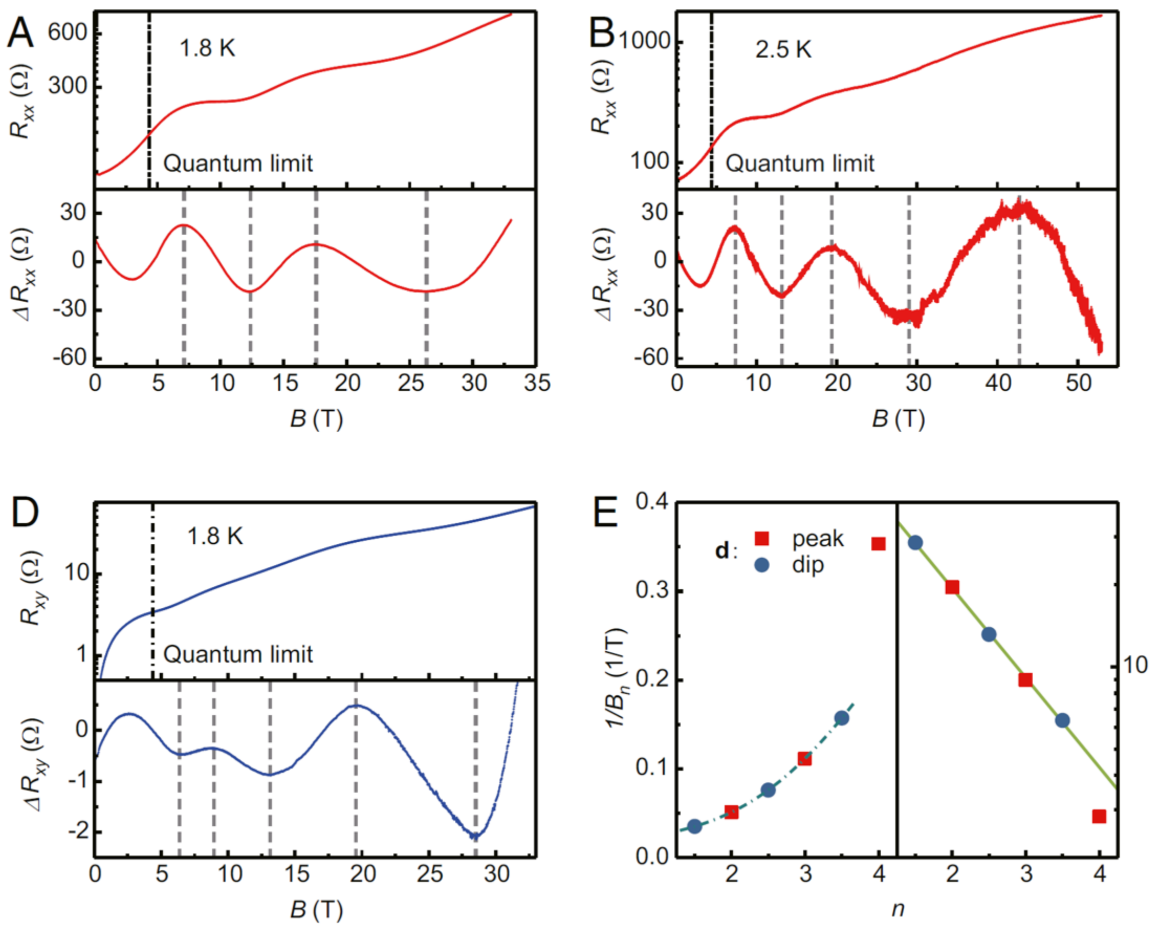
Peak values of B



Log-periodic oscillations in Bi (B > 10T)

Log-periodic oscillations in Te

曾长淦研究组碲中霍尔电阻和纵向电阻呈现出离散标度不变性。



ferent energies approaching the Fermi energy continuously. The as-induced scattering between free carriers and quasi-bound states thus influences the transport properties (21), manifesting as the log-periodic oscillations for the measured MR and Hall data.

Here we would like to attribute the observed log-periodic oscillations in Te crystals to the DSI after excluding another

21, the semiclassical quantization condition leads to the DSI

with $\lambda = e^{2\pi/s_0}$. Here $s_0 = \sqrt{(Z\alpha)^2 - \kappa^2}$, which can be simplified

to $s_0 = \sqrt{\alpha^2 - 1}$ when we consider the charge number $Z = 1$ and the lowest angular momentum channel with $\kappa = \pm 1$ (21). Thus, the fine-structure constant (α) is calculated to be ~7.5, far exceeding 1/137 due to the small Fermi velocity of Te crystals. According to $\alpha = e^2/4\pi\varepsilon_0\hbar v_F$, the Fermi velocity v_F is calculated to be ~2.9×10^5 m/s, which is comparable to the value (~1.9×10^5 m/s) estimated from the calculated band structure. The small v_F ensures the supercritical condition, promoting the formation of quasi-bound states with DSI (21).

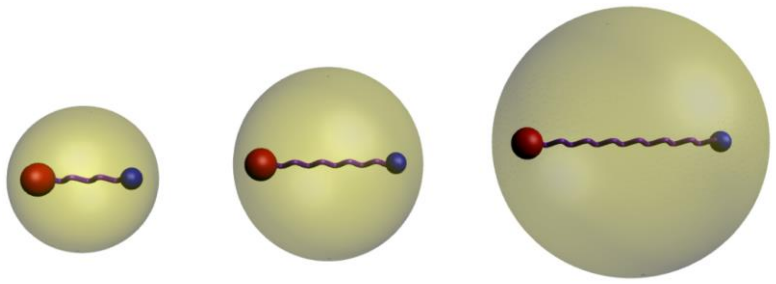
21. H. Wang *et al.*, Discovery of log-periodic oscillations in ultraquantum topological materials. *Sci. Adv.* **4**, eaau5096 (2018).
22. H. Wang *et al.*, Log-periodic quantum magneto-oscillations and discrete-scale invariance in topological material $HfTe_5$. *Natl. Sci. Rev.* **6**, 914–920 (2019).
23. H. Liu, H. Jiang, Z. Wang, R. Joynt, X. Xie, Discrete scale invariance in topological semimetals. arXiv:1807.02459 (6 July 2018).

N. Zhang, *et al.*, and C. Zeng*, PNAS.2002913117 (2020).

Conclusion

1. Dirac particles with supercritical Coulomb attraction = two-body quasi-bound states with Discrete Scale Invariance.

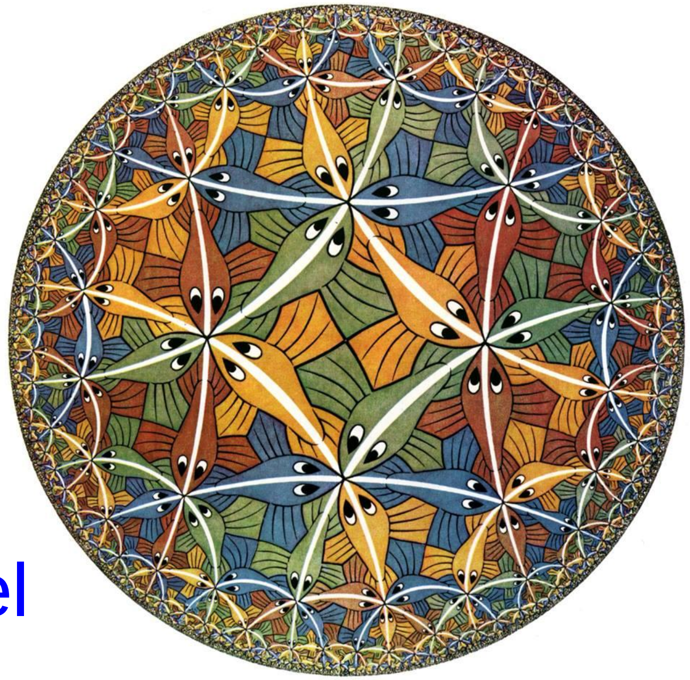
2. Under a magnetic field, the resonant scatterings between mobile carrier and quasi-bound states gives rise to log-B periodic oscillation beyond the quantum limit.



Science and Art

Where the world ceases to be the stage for personal hopes and desires, where we, as free beings, behold it in wonder,
to question and to contemplate,
there we enter the realm of art and science.
Common to both is the devotion to something beyond the personal, removed from the arbitrary.

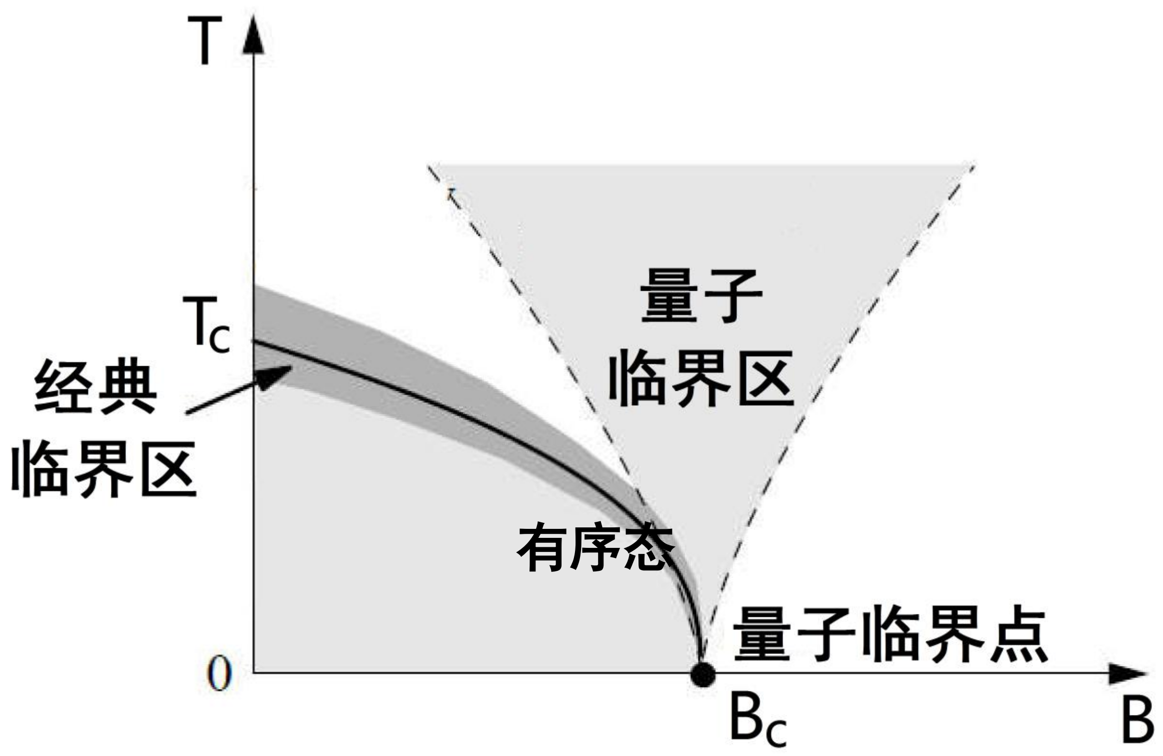
--Albert Einstein



"Circle Limit III" M. C. Escher
Poincaré conformal disk model

Rare event effect on Quantum Phase transition: Quantum Griffiths Singularity

Introduction to Quantum Phase Transition



T= 0 :

关联长度 $\xi \sim |B - B_c|^{-\nu}$

关联时间 $\xi_\tau \sim \xi^z$

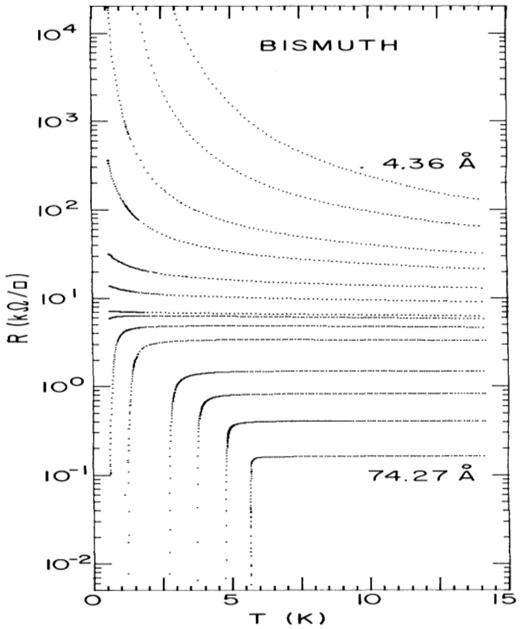
T > 0 温度特征尺度有限尺寸标度律

电阻 $R = R_0 \Phi\left(\frac{L_\tau}{\xi_\tau}\right) = R_0 \widetilde{\Phi}\left(\frac{|B - B_c|^{z \cdot \nu}}{T}\right)$

S. L. Sondhi, S. M. Girvin, J. P. Carini, D. Shahar, RMP (1997).

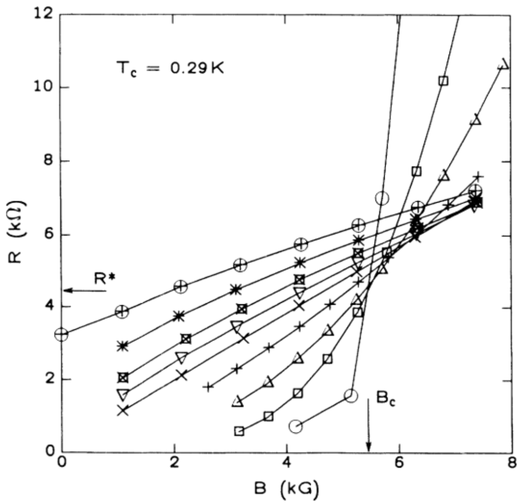
Background on SIT/SMT

a-Bismuth薄膜



A. Goldman 1989

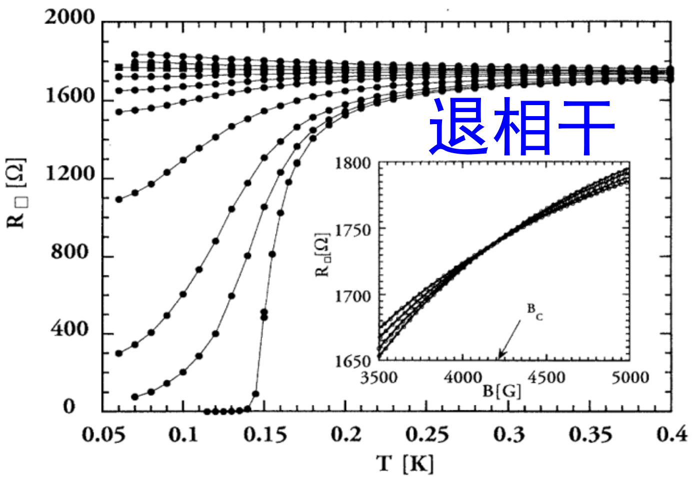
a-InOx薄膜



AF Hebard 1990

玻色型局域化模型

MPA Fisher 1990



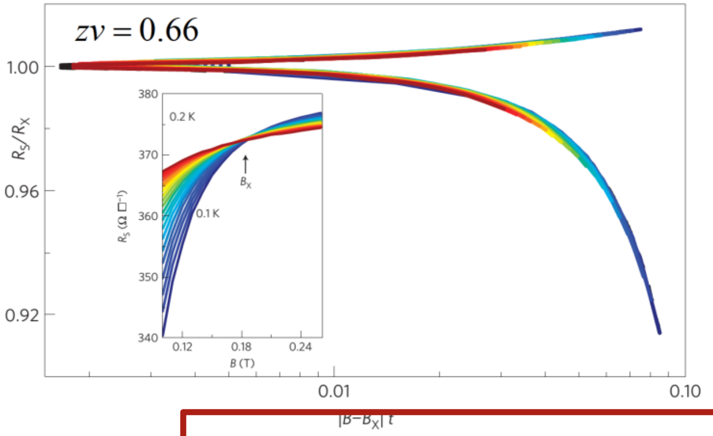
A. Kapitulnik 1995

2014年Buckley奖（美国凝聚态物理最高奖）表彰他们对超导-绝缘体/金属相变的贡献。

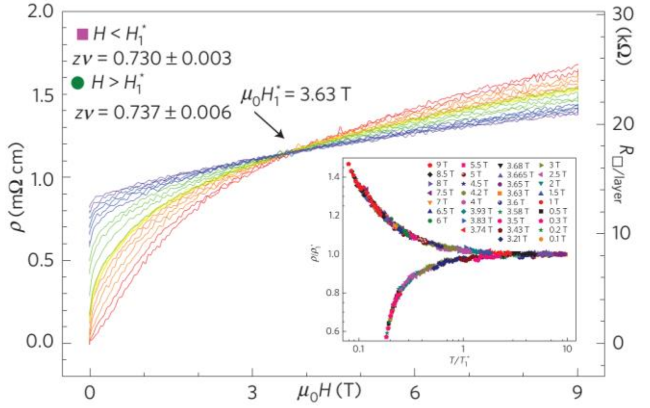
传统观点：一个交点，一个临界指数

Background on SIT/SMT

LaTiO3/SrTiO3界面

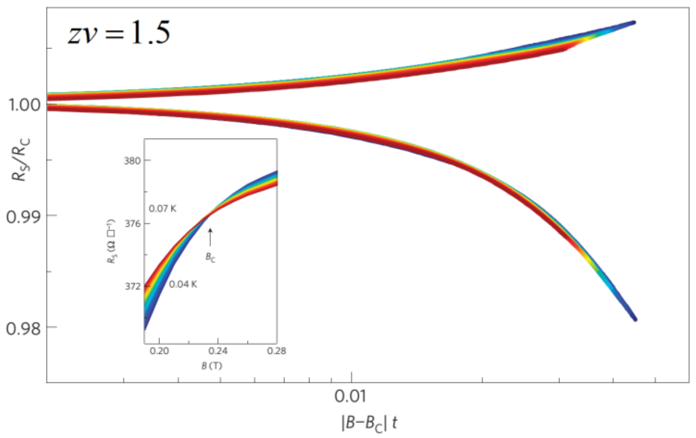


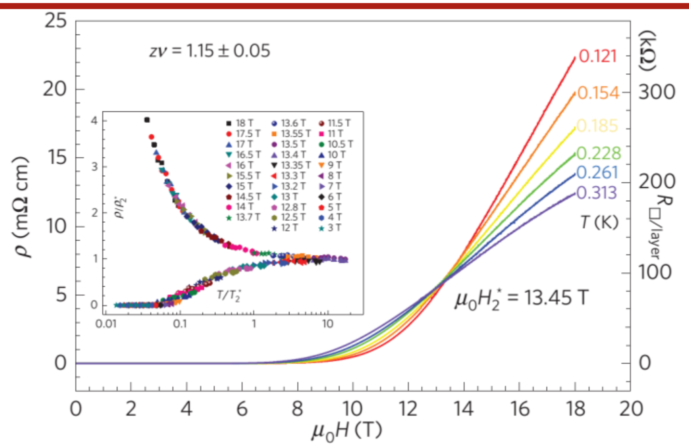
La2-xSrxCuO4薄膜



超导-金属相变是普适的么？

过去发现：两个交点，两个临界指数



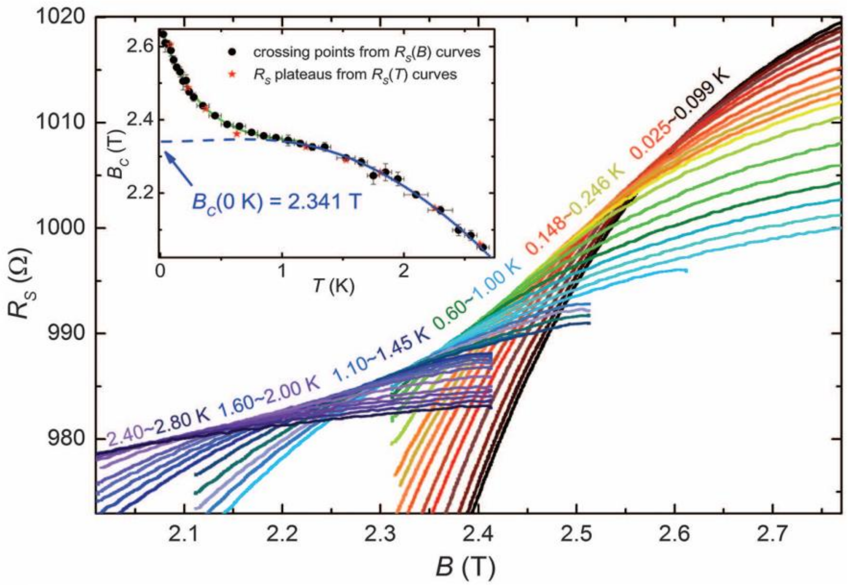


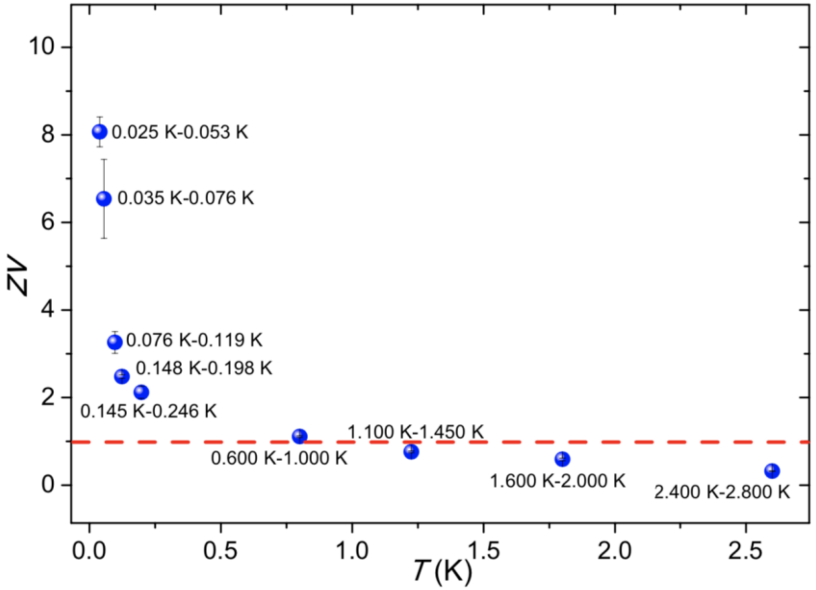
是否存在新的理论框架来解释实验？

J. Lesueur *et al*., Nat. Mat. **12**, 542 (2013).

D. Popovic *et al*., Nat. Phy. **10**, 437 (2014).

SMT in Ga thin film





镓薄膜中的超导–金属相变

极低温段临界电阻上翘，明显偏离平均场结果。

多个交点，多个临界指数，与原理论严重不符！

$z \cdot \nu$取值不断变化，呈现发散趋势，最大值8.07。

Rare event effect on QPT: QGS in SMT

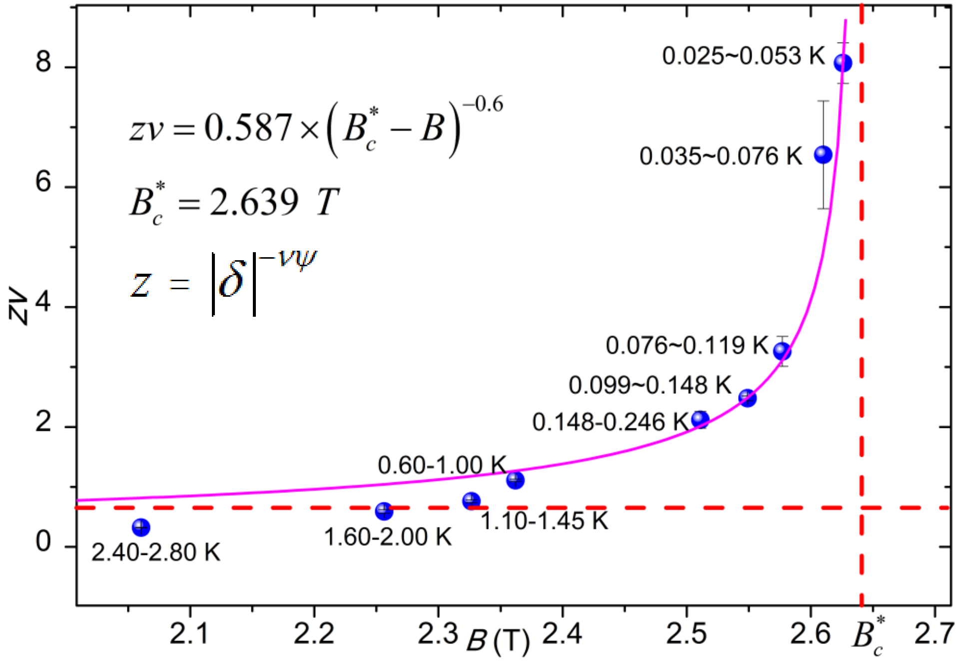
U(1)序+欧姆型耗散的体系的低能特征能标：

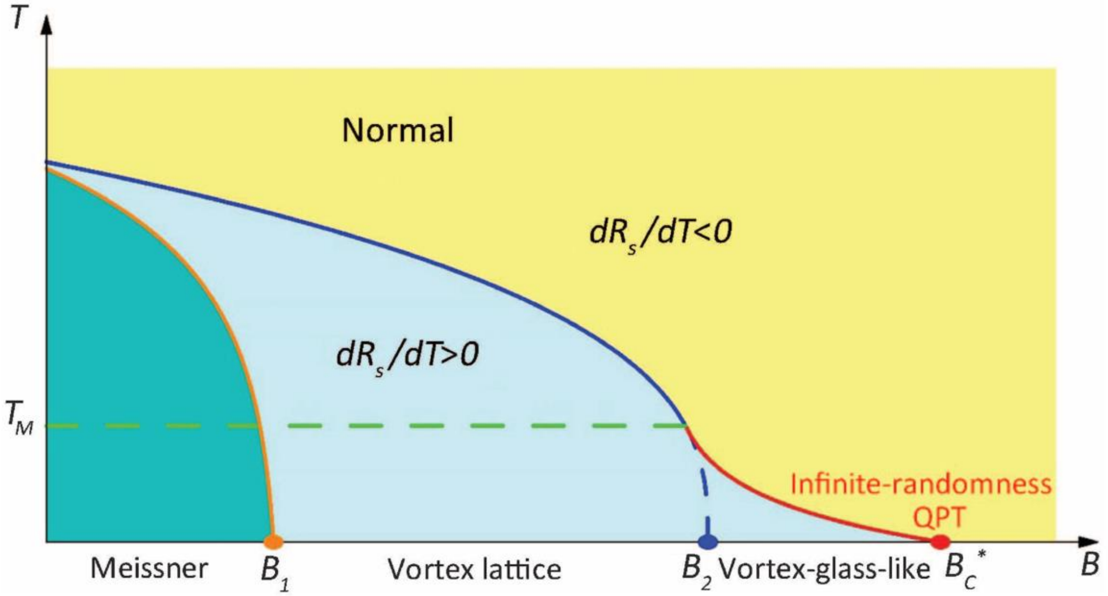
$$\delta E \propto \exp\left(-c_2 L^d\right)$$

量子Griffiths奇异性

普适性

理论分析发现存在杂质效应和欧姆型耗散时，超导金属相变与随机横场Ising模型属于同一个普适类，$z \cdot \nu$随磁场变化满足量子Griffiths奇异性。





X. Xing#, H. Zhang#, H. Fu#, H. Liu#, *et al.*, Science **350**, 542 (2015).

Fluctuation effect on Ising superconductivity in Pb

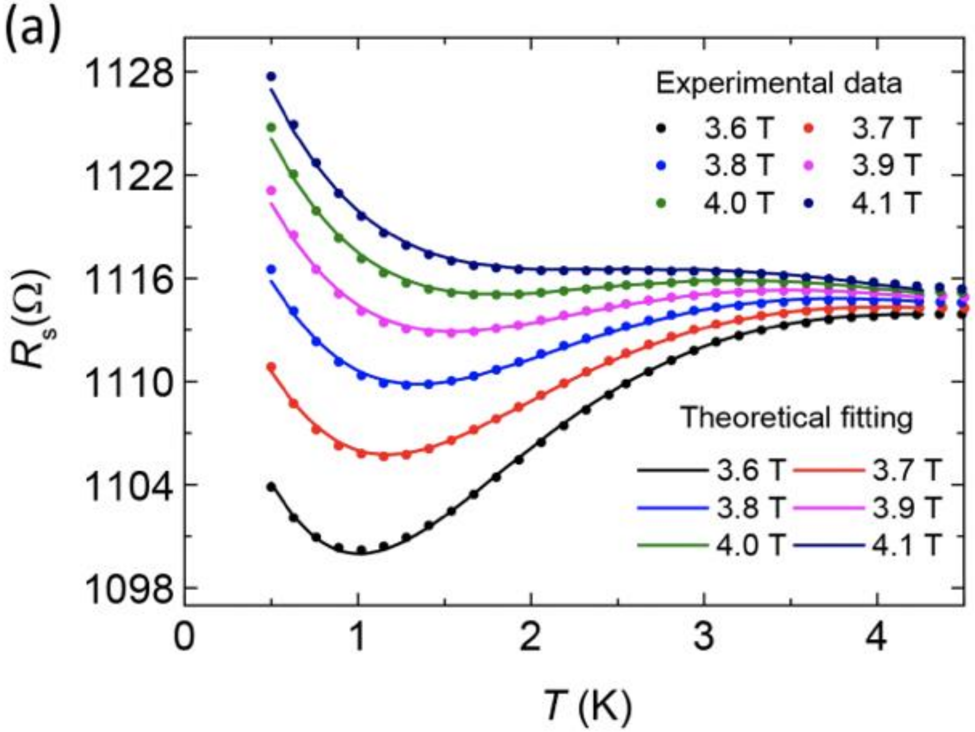
We consider the influence of SOI on the Aslamazov-Larkin (AL) term and the MT-type terms

$$\sigma = \sigma_n + \frac{e^2}{\pi^2 \hbar}\left[\alpha I_\alpha(b,t) + \beta I_\beta(b,t)\right] + C\left[\exp\left(-\frac{\Delta}{k_B T}\right) - 1\right]$$

$$I_\alpha(b,t) = \ln\frac{r}{b} - \frac{1}{2r} - \psi(r) \qquad I_\beta(b,t) = r\psi'(r) - \frac{1}{2r} - 1$$

Δ is the the local superconducting pairing strength)

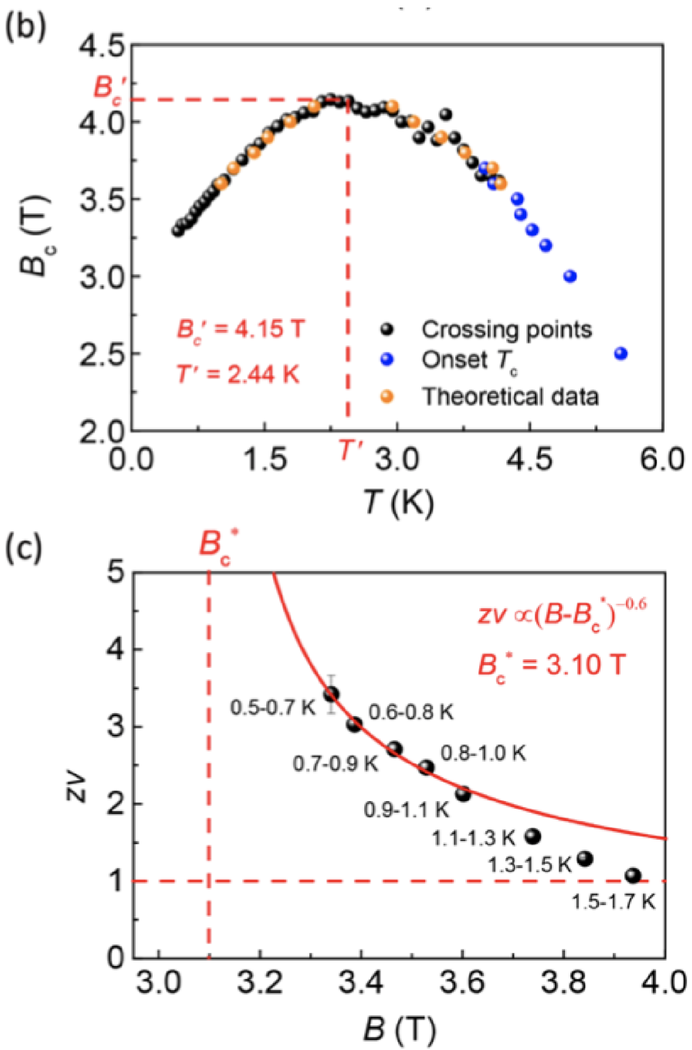
$\psi(r)$is the digamma function

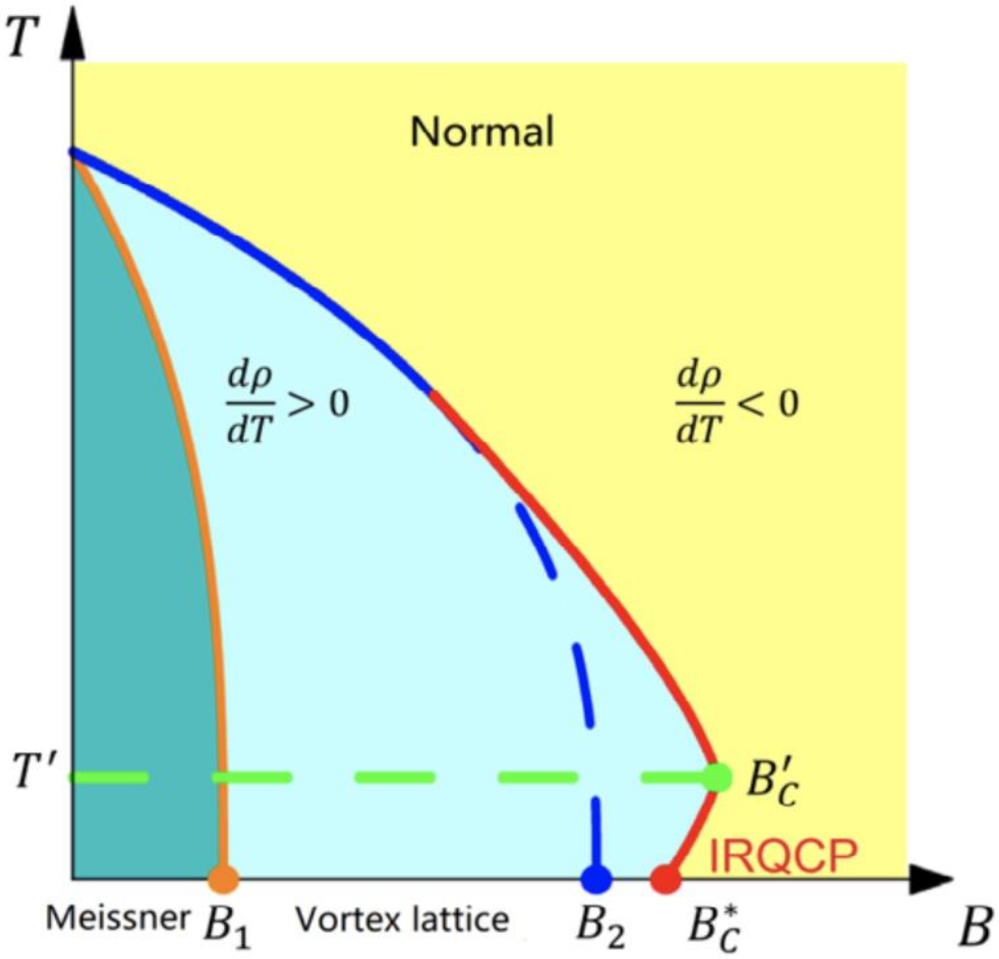


B (T)	α	β	C (Ω^{-1})	Δ/k_B (K)
3.6	0.518	-0.233	9.71×10^{-5}	10.2
3.7	0.518	-0.218	9.47×10^{-5}	10.3
3.8	0.525	-0.205	9.33×10^{-5}	10.2
3.9	0.527	-0.194	9.09×10^{-5}	10.2
4.0	0.528	-0.195	8.72×10^{-5}	10.3
4.1	0.515	-0.171	8.27×10^{-5}	10.3

Y. Liu, ..., H. Liu*, et al. J. Wang*, Nat. Commun. 10, 3633 (2019).

Fluctuation effect on type-I Ising superconductivity in Pb

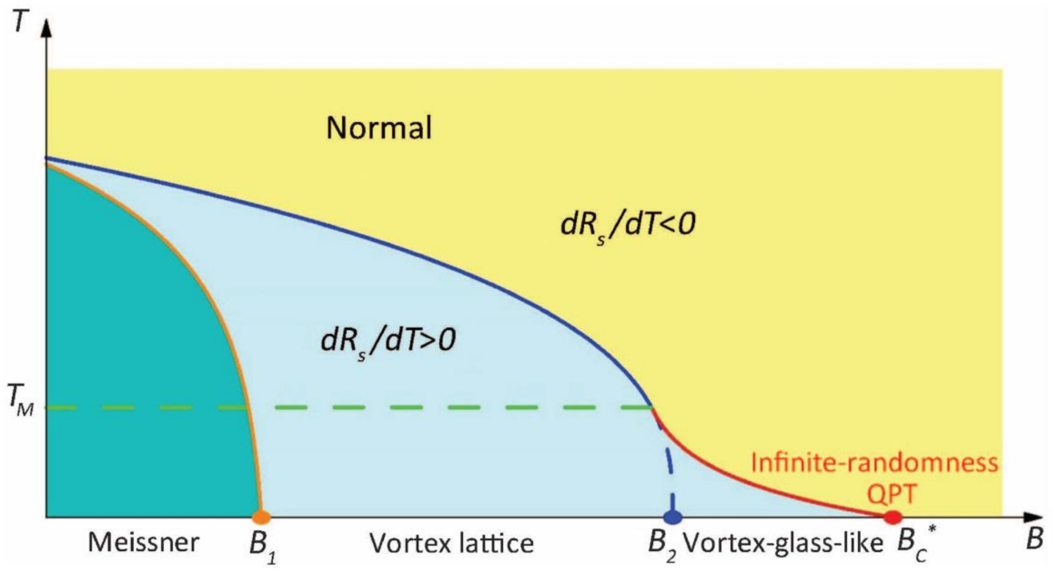


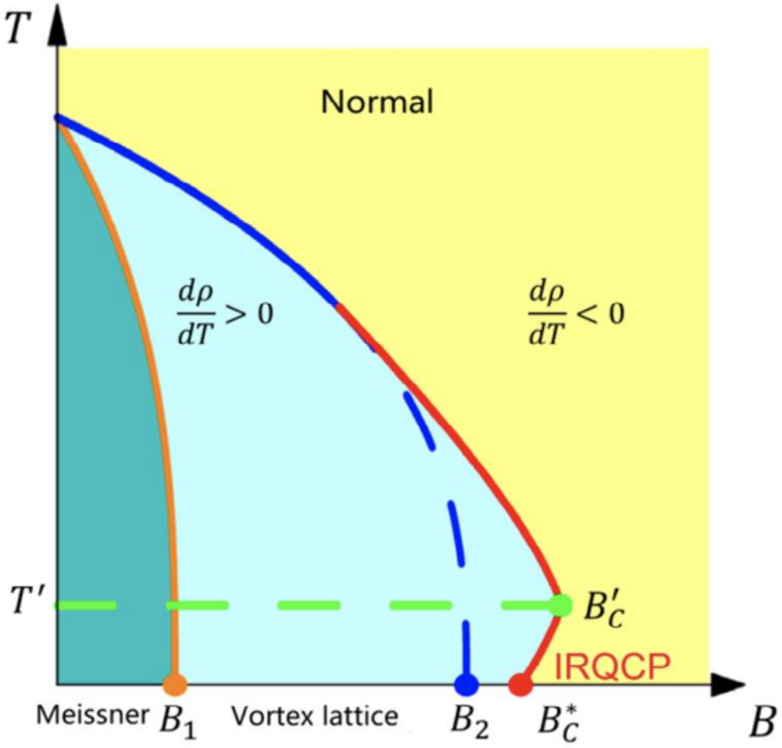


Anomalous quantum Griffiths singularity in 4-ML Pb films due to pronounced fluctuation effect.

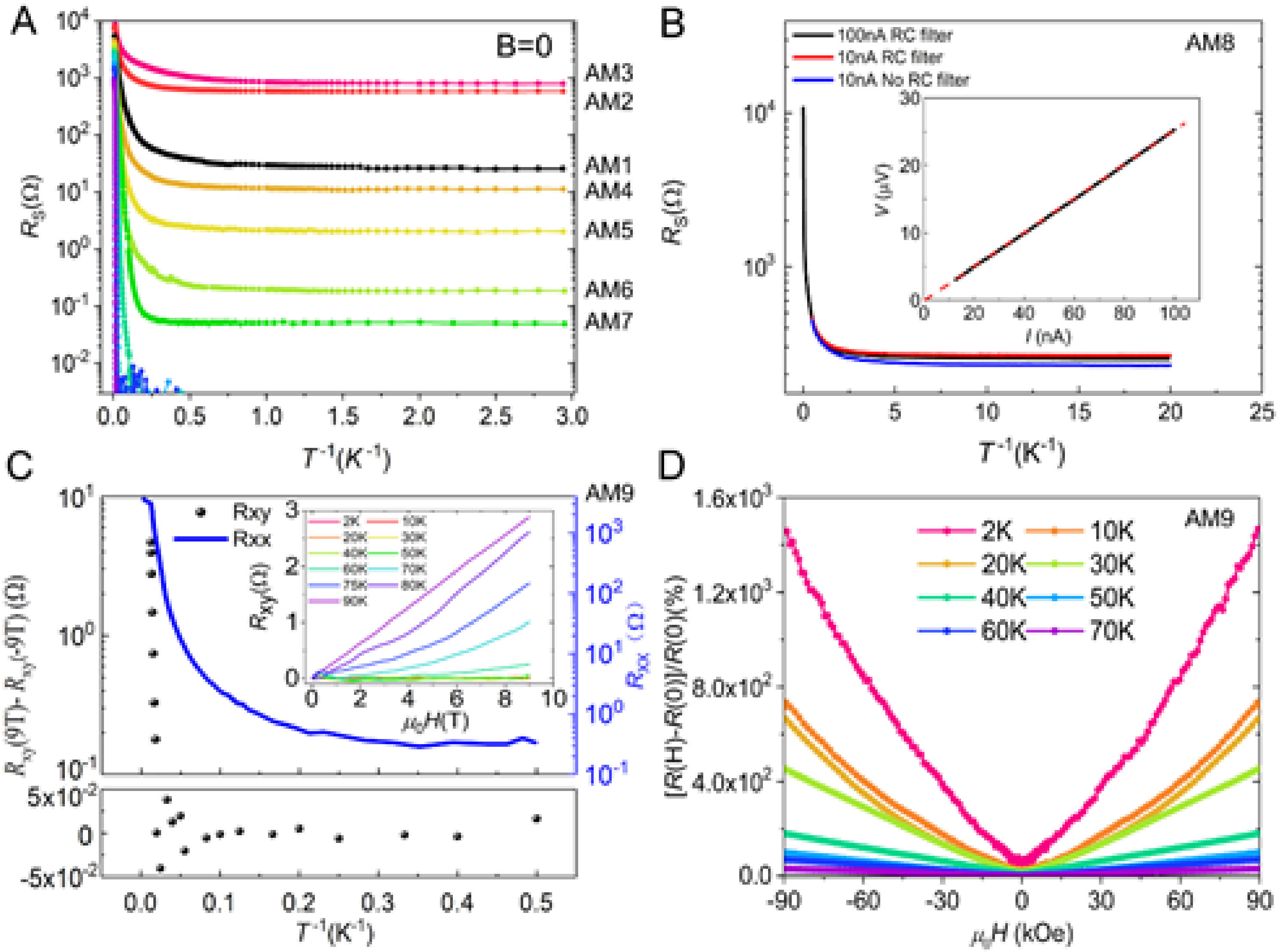
Y. Liu, ..., H. Liu*, *et al.*, J. Wang*, Nat. Commun. **10**, 3633 (2019).

Rare event can dramatically change the quantum phase transition, and gives rise to Quantum Griffiths singularity, a novel type of universality class.



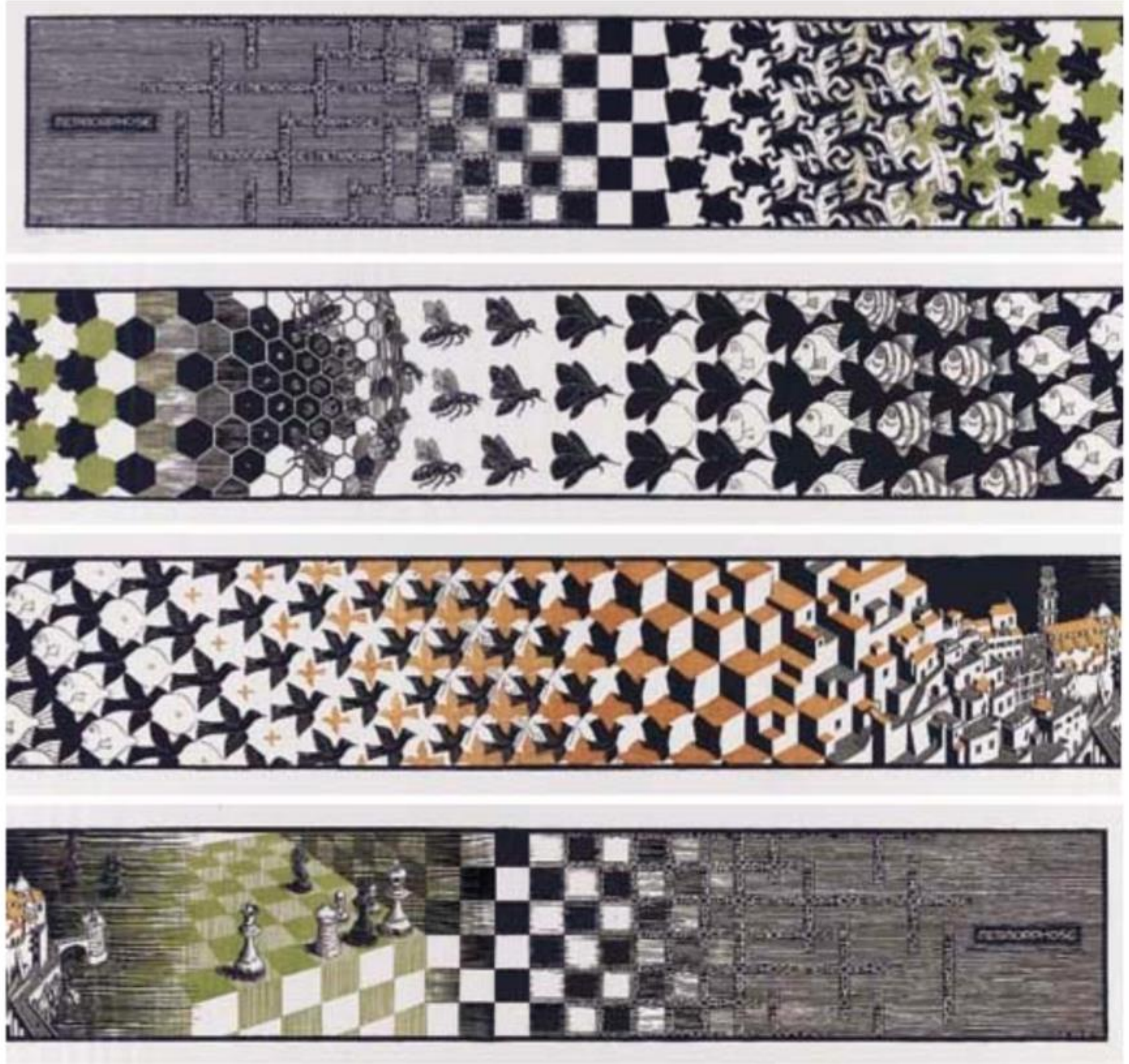


Puzzles and progress: relation of rare event to “anomalous quantum metal”



C. Yang, et al., Science 366, 1505 (2019).

Anomalous is normal

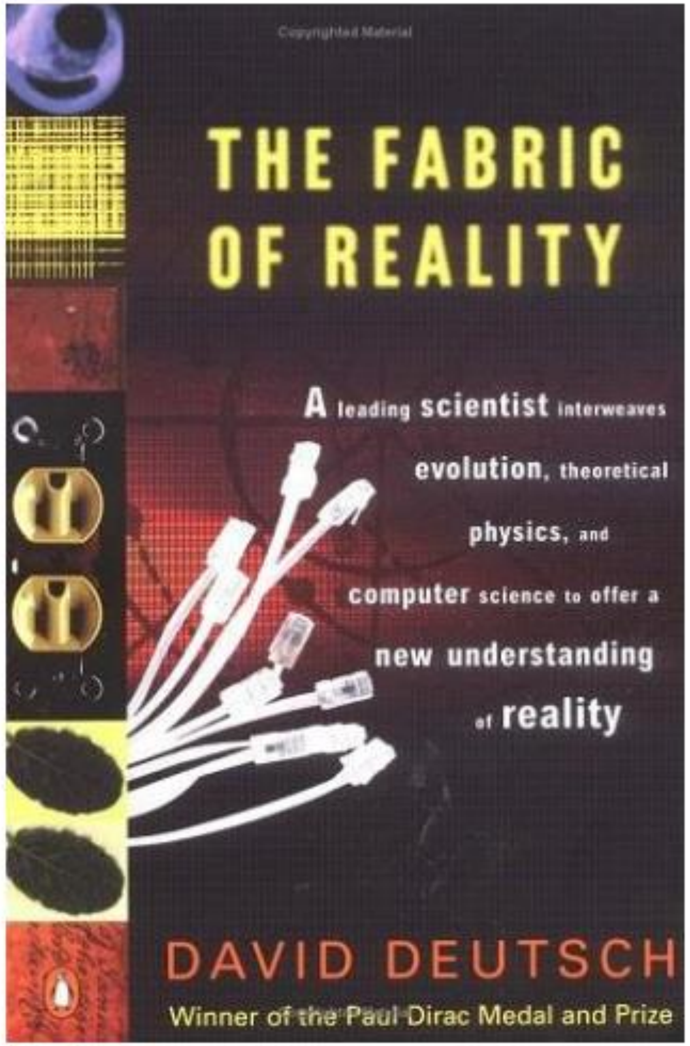


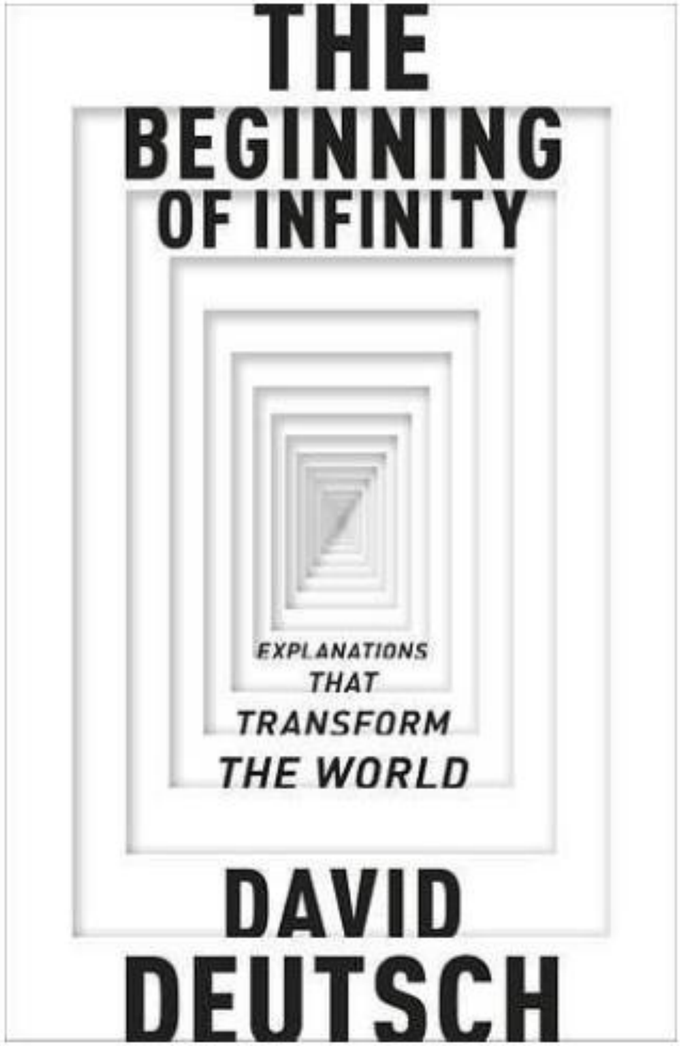
"Metamorphosis II"
M. C. Escher

The progress has no necessary end, it is the beginning of infinity.

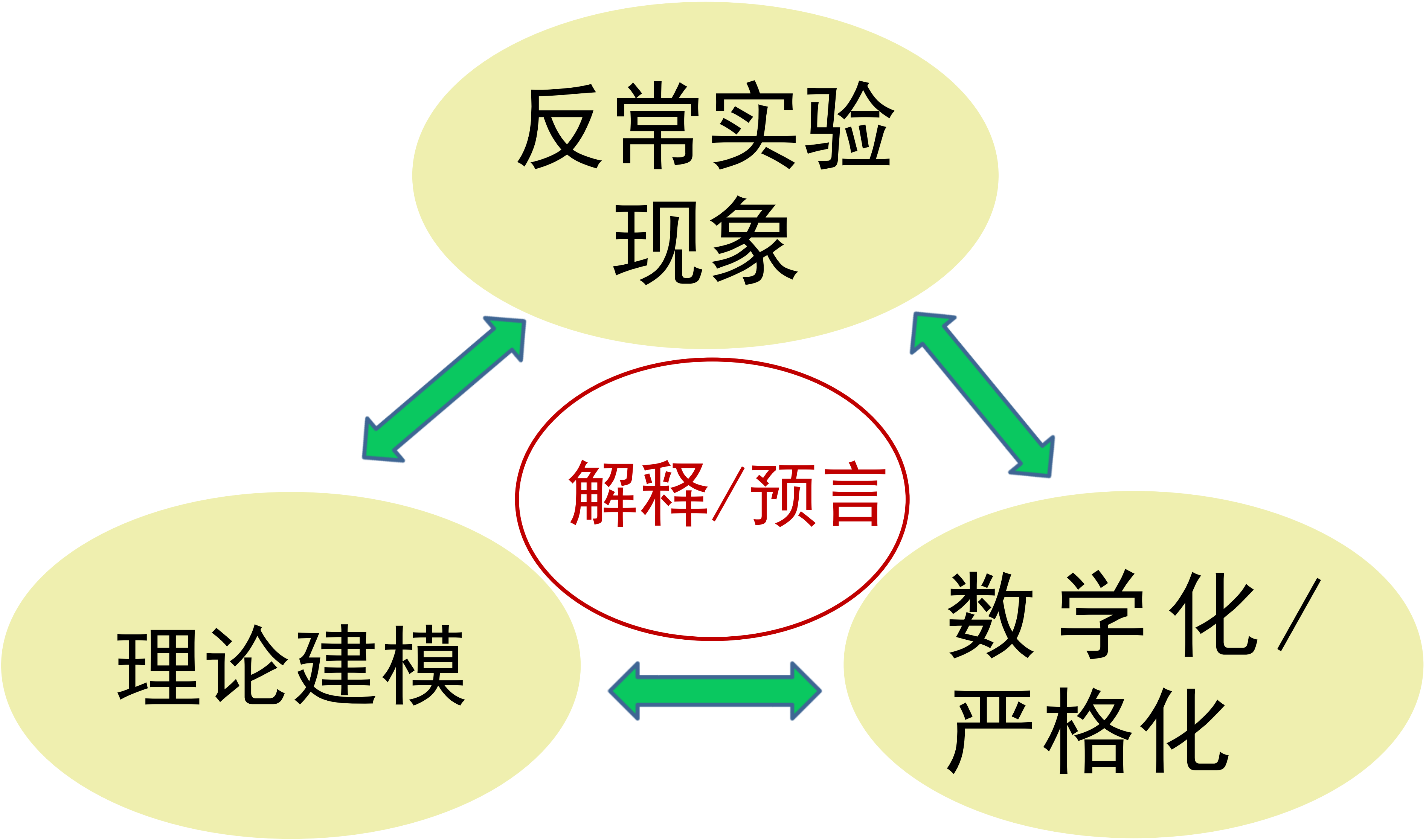
--David Deutsch

In retrospect to Thomas Kuhn "paradigm shift"





黄金三角



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Thank you !